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Strong coupling limit of Bethe Ansatz equations

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Abstract

We develop a method to analyze the strong coupling limit of the Bethe ansatz equations supposed to give the spectrum of anomalous dimensions of the planar $\mathcal{N} = 4$ gauge theory. This method is specially adapted for the three rank-one sectors, $su(2)$, $su(1|1)$ and $sl(2)$, where we study the highest excited states, for the first two sectors, and the dimension of the twist-two operator for the last one. We use the elliptic parametrization of the Bethe ansatz variables, which degenerates to a hyperbolic one in the strong coupling limit. We consider the equations for the highest excited states in the $su(2)$ and $su(1|1)$ sectors and for the state corresponding to the twist-two operator in the $sl(2)$ sector, both without and with the dressing kernel. In most of the cases, we are able to solve for the densities of magnons at the leading order. In all the cases, we obtain the leading order of the anomalous dimensions, which reproduce, with an exception, the results obtained previously by numerical or analytical means.

Contents

1 Introduction

There is mounting evidence that the planar $\mathcal{N} = 4$ SYM gauge theory is integrable [1, 2, 3] up to higher orders in perturbation theory, nourishing the hope to prove by this mean its equivalence with the string theory on $AdS_5 \times S^5$ [4, 5, 6]. Under the assumption of integrability, the spectrum of the anomalous dimensions of the planar $\mathcal{N} = 4$ SYM gauge theory is encoded, at least for states with large number of operators, in the Bethe ansatz equations. The S-matrix which is behind these equations is fixed by symmetry [7, 8], up to a scalar factor which is supposed to obey a crossing symmetry relation [9]. At strong coupling, the scalar factor should allow to reproduce the string theory results, and its first two orders in the strong coupling expansion were fixed in this way [10, 11]. Few months ago, based on previous work of Beisert, Hernández and López [12], Beisert, Eden and Staudacher [13] proposed an expression for the scalar factor which is crossing symmetric and reproduces both the strong coupling, string results and the perturbative results for the anomalous dimensions in the gauge theory. An efficient way of testing the predictions of the Bethe ansatz equations proved to be via the anomalous dimension of the twist-two operator. In the regime of large Lorentz spin S, this quantity scales logarithmically

$$\Delta - S = f(g) \ln S + \dots \; . \tag{1.1}$$

The universal scaling function for $\mathcal{N} = 4$ SYM up to third loop order was extracted [14] from perturbative three-loop computation in QCD [15]. The same quantity appears in the iterative structure [16] of the multi-gluon amplitudes of the supersymmetric gauge theory, and it was computed up to three loop order in [17], and, by an impressing effort, up to four loop order in [18, 19]. The four loop result agrees with the prediction based on the proposal [13]. On the other hand, at strong coupling the universal scaling function $f(g)$ is predicted to behave as [20, 21]

$$f(g) = 4g - \frac{3 \ln 2}{\pi} + \dots \; . \tag{1.2}$$

After the proposal [13] was made, several papers [22, 23, 24] were devoted to derive from it the strong coupling expansion (1.2). While the numerical work [23] easily reproduces the behavior (1.2) and even predicts the next term in the expansion, the analytical treatment proved to be much more difficult. Up to now, only the first term was obtained analytically [22, 24].

One of the aspects which deserve further attention is to understand the origin of the dressing phase [13] and to test its validity. It is believed that the dressing phase comes from a non-trivial structure of the vacuum. Very recently, a structure similar to the dressing phase was obtained [25], via the nested Bethe ansatz, for one of the non-trivial "vacuum" states in one of the sectors which are not of rank one.

The scope of this paper is to develop a method of analyzing the Bethe ansatz equations at strong coupling. Our analysis is based on the formulation of the Bethe equations as integral equations, and it works most simply on, although it is not restricted to, states as the antiferromagnetic state in the $su(2)$ sector, on the most excited state in the $su(1|1)$ sector, or for the finite-twist operators, in the $sl(2)$ sector. Some of this cases were already studied numerically or analytically; here we aim to a unitary treatment and hope that this method will be useful to develop systematically a perturbative expansion in the coupling constant $1/g$. This perturbative expansion is clearly more difficult to perform than the weak coupling expansion, around $g = 0$.

Our starting point is the elliptic parametrization of the variables u, $x^{\pm}$ appearing in the Bethe equations. The elliptic parametrization appeared for the first time in the work of Janik [9]. The one we use here is related by a Gauss-Landen transformation to the one in [9], and it appeared independently in [8]. The elliptic modulus is defined by

$$\frac{k'}{k} = \frac{1}{4g} \equiv \epsilon \; . \tag{1.3}$$

The elliptic parametrization degenerates into a hyperbolic one in the limit $g \to \infty$, and it offers the natural variables to work with in this limit. In particular, it becomes transparent which are the important regimes to be considered, and which were recently discussed in [26].

As it became clear from the numerical solutions, the points $u = \pm 2g$ become singular for large g and they split the real axis u into two regions, with $|u|/2g$ less or larger than unity, where two separate hyperbolic parametrizations apply.[1] In terms of the momentum p, these regions correspond to a finite value of p, for $|u|/2g < 1$, or the region of "giant magnons", in the language of [26], and momenta of the order $p \sim 1/g$, or $|u|/2g > 1$, correspond to the "plane-wave" limit [27]. What is more difficult to see is what happens exactly at $|u|/2g = 1$ and in the vicinity. These points are not taken into account properly by the hyperbolic parametrization. In fact, they correspond to momenta of the order $p \sim g^{-1/2}$, or to the "near-flat space" region, again in the terminology of [26]. In some cases all the solutions of the Bethe equations concentrate in this region. It is possible to use the elliptic parametrization to obtain an expansion around this points,[2]and it is clear that the expansion in this region involves powers of $g^{-1/2}$, or equivalently $1/\lambda^{1/4}$.

In order to take the strong coupling limit of the dressing kernel [13] we have several options. The first one is to take the large coupling limit term by term in the series defining the dressing phase around $g = \infty$ [12, 13]. For the giant magnon and plane-wave limits this works, and this is probably the most straightforward way to obtaind the leading orders. In the near-flat space limit, however, as it was noticed in [26], all the terms of the series contribute to the leading order. A way to circumvent this difficulty is to use the representation of the dressing kernel given by the "magic formula" of [13]. This representation was already used in [24], but this such a procedure is of limited interest, since it is difficult to compute the higher orders in $1/g$, and, in particular, it is difficult to use it in the regime of near-flat space. Here, we give another integral representation, which is, we believe, better adapted to this particular regime. We would like to emphasize that, although the results for the leading order of the anomalous dimensions can be obtained without being particularly careful about the near-pane wave regime, the corrections are crucially determined by it. The representation we propose for the dressing kernel is

$$K_d(u,u') = 4\sum_{n\geq 1}\int_{-\infty}^{\infty} dv\; K_-^{1,n}(u,v)\;\left(K_+^{n,-1}(v,u') - K_+^{n,1}(v,u')\right)\,, \tag{1.4}$$

where the kernels $K_\pm(u,u')$ are the odd/even part of the basic $su(1|1)$ kernel $K(u,u')$, the inverse Fourier transform counterparts of the kernels $K_{0,1}(t,t')$ in the BES decomposition [13]. The upper indices $K_\pm^{m,n}(u,u')$ mean essentially that the variables u and u' are defined by elliptic functions of modulus

$$\frac{k'_n}{k_n} = \frac{n}{4g}\,, \qquad \frac{k'_m}{k_m} = \frac{m}{4g}\,. \tag{1.5}$$

The sum in (1.4) resembles to a sum over intermediate states; in particular, the energy

[1] The two hyperbolic parametrizations are related by the transformation $s \to K - s$, where K is the real period of the elliptic function.

[2] In this case, the parametrization is obtained by shifting $s \to \pm K/2 + s$.

of these "intermediate states" is of the type [30]

$$E_n \sim \sqrt{1 + \frac{16g^2}{n^2} \sin^2 p/2} \,. \tag{1.6}$$

In the large g limit, the sum over n can be taken with the help of the Euler-Maclaurin formula and the integrals over the variables v and $z = n/4g$ are simple enough to be done.

Turning to the explicit results, we obtain here the leading order result for anomalous dimensions of the highest excited state in the $su(2)$ and $su(1|1)$ sectors, both for the Bethe ansatz with the Beisert, Hernández, Lopez/Beisert, Eden, Staudacher (BHL/BES) phase, and without it. Even if the old kernel, without the dressing phase, is by now only of limited interest, we can still ask the question whether the $su(1|1)$ sector arises from an effective model, in the same way the $su(2)$ sector appears from the reduction of the Hubbard model to the half-filling. The large g limit of the $su(1|1)$ sector should give a hint for the underlying degrees of freedom. In the $su(2)$ sector, which was solved in [31, 32] the degrees of freedom are magnons, which become free as $g \to \infty$. Our result in the $su(1|1)$ sector shows that these degrees of freedom do not correspond to a free system, and they are rather difficult to characterize. Two thirds of the excitations are of the type "giant magnon", with finite momenta, and one third are of the type "plane wave", with momenta of order $1/g$, see also the numerical results of [33]. We find for the energy without the dressing factor

$$E_{su(1|1)} = \frac{8 \ln 2}{\pi} gL \,, \qquad E_{su(2)} = \frac{4}{\pi} gL \,. \tag{1.7}$$

The $su(2)$ energy is the $g \to \infty$ limit of the exact result [31, 32]. When taking into account the BHL/BES phase, almost all the solutions of the Bethe equations, both for the $su(2)$ and $su(1|1)$ sector, fall into the region "near-flat space", with momenta of order $1/\sqrt{2}$. The integral equations are able to reproduce the leading result for the anomalous dimensions of the corresponding states, which were also obtained numerically and analytically in [33, 34].

$$E^d_{su(1|1)} = \sqrt{2\pi g} L \,, \qquad E^d_{su(2)} = \sqrt{\frac{\pi g}{2}} L \,. \tag{1.8}$$

For the $su(2)$ state, the value we obtain is $\sqrt{2}$ times less than the one obtained in [34].

We have also analyzed the strong coupling limit of the equation derived by Eden and Staudacher (ES) [35] for the anomalous dimension of the twist-two operator without and with the dressing phase (BES equation). Similarly to the approaches [23, 22], we obtain that the strong coupling limit of the ES equation is pathological. The BES equation has a better behavior, and we are able to reproduce the leading term of the density obtained recently by Alday *et al.* [24] by using the Fourier space representation,

$$\sigma^<(u) = \frac{1}{4\pi g^2} \qquad \text{for} \qquad |u| < 2g \,, \tag{1.9}$$

$$\sigma^>(u) = \frac{1}{4\pi g^2}(1 - \cosh s/2) \qquad \text{for} \qquad u = 2g \coth s \,, \quad |u| > 2g \,.$$

For the next order in the density, an important fraction of the roots lie near the points $u \simeq \pm 2g$. We show that without properly considering the scattering of the magnos in this region, the density is non-normalizable. We leave the fine analysis of the near-flat space region for future work.

The paper is organized as follows: in section 2 we write down the integral equations corresponding to the three sectors, in section 3 we explicit the elliptic parametrization and we take the strong coupling limit of the building block, the $su(1|1)$ kernel. In section 4 we compute the energy of the highest states for the $su(2)$ and $su(1|1)$ without the dressing kernel and discuss the leading order solution of the Eden-Staudacher equation. In section 5 we give an integral representation of the dressing kernel and we check that it reproduces the already known results at strong coupling. We also give the strong coupling limit of the AFS phase [10]. The integral equations with the dressing phase equations are considered in section 6.

2 Bethe ansatz equations in integral form

If we consider highly excited states, with a large number of magnons, the Bethe ansatz equations can be solved by transforming them into integral equations. These integral equations are particularly simple when the number of magnons is maximal, such that there are no holes in the magnon distribution. This is the case of the highest excited state in the $su(1|1)$ sector and the antiferromagnetic state in the $su(2)$ sector. By an appropriate transformation, the leading twist operator, belonging to the $sl(2)$ sector, can be put in a similar form[3]

The equations in the three rank-one sectors can be written in the compact form [36, 7]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_l^M \frac{1 - g^2/x_k^+ x_l^-}{1 - g^2/x_l^+ x_k^-} \left(\frac{x_k^+ - x_l^-}{x_k^- - x_l^+}\right)^\eta e^{i\theta(u_k,u_l)} , \tag{2.1}$$

where $\eta = 1, 0, -1$ for $su(2)$, $su(1|1)$ and $sl(2)$ respectively. The variables $x^\pm$ are defined by

$$x(u) = \frac{1}{2}u\left(1 + \sqrt{1 - 4g^2/u^2}\right) , \quad x^\pm = x(u \pm i/2) .$$

It is often convenient to work with the momentum p, the physical variable of the magnon, defined by

$$u(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1 + 16g^2\sin^2\frac{p}{2}} , \qquad \text{or} \quad x^+/x^- = e^{ip} . \tag{2.2}$$

We are going to consider both the case where the phase $\theta(u_k, u_l)$ is zero, and the BHL/BES dressing phase [13].

Generically, the Bethe equations for the rank-one sectors can be written as

$$e^{ip_k L} = \prod_{l=1}^M e^{i\varphi(u_k,u_l)} , \tag{2.3}$$

[3]The integral equation for the highest state in the $su(1|1)$ sector was written down by one of us and M. Staudacher, [37] and in [40], the equation for the antiferromagnetic state in the $su(2)$ sector was solved in [31], and the equation in $sl(2)$ sector, or the Eden-Staudacher equation, was derived in [35].

where $\varphi(u_k, u_l)$ is the scattering phase for two magnons. Let us exemplify the derivation of the integral equation on the $su(1|1)$ case, which is the simplest one. We consider for convenience L odd. The most excited state will contain a number of $L-1$ magnons and formally we can add a L-th magnon with momentum $p = 0$ in the equations, such that $M = L$. When $g = 0$, the equations (2.1) correspond to a free fermion system, with the occupation numbers

$$p_k^{(0)} = 2\pi k/L \,, \qquad k = -\frac{L-1}{2}, \ldots, \frac{L-1}{2} \,. \tag{2.4}$$

When g increase, the momenta start to evolve, according to the Bethe ansatz equations (2.1)

$$\frac{p_k}{2\pi} = \frac{k}{L} + \frac{1}{2\pi L} \sum_{l=1}^{M} \varphi(u_k, u_l) \,. \tag{2.5}$$

When the length of the chain is very large, we can take the continuum limit. We introduce the variable $t = k/L$ and the density of rapidities $\rho(u) = -dt/du$. The derivative with respect to u of the equation (2.5) gives

$$\rho(u) = -\frac{1}{2\pi}\frac{dp}{du} + \int_{-\infty}^{\infty} du' \; K(u,u') \; \rho(u') \,, \tag{2.6}$$

where the kernel is the derivative of the scattering phase phase

$$K(u,u') = \frac{1}{2\pi}\frac{d}{du}\, \varphi(u,u') \,. \tag{2.7}$$

The derivation of the $su(2)$ equation is similar, with the difference that the maximum number of magnon is $L/2$ instead of L. This derivation follows closely the analysis of the antiferromagnetic state in the XXX model [38].

In the following, we are going to denote by $K(u,u')$ the kernel corresponding to the $su(1|1)$ case without the dressing phase. Another building block is the $su(2)$ kernel, $K_{su(2)}(u,u')$, and the third one is the dressing kernel, corresponding to the dressing phase $\theta(u,u')$:

$$\begin{aligned} K(u,u') &= \frac{1}{2\pi i}\frac{d}{du}\left(\ln\left(1-\frac{g^2}{x^+(u)x^-(u')}\right) - \ln\left(1-\frac{g^2}{x^-(u)x^+(u')}\right)\right) && (2.8)\\ K_{su(2)}(u,u') &= \frac{1}{2\pi i}\frac{d}{du}\ln\left(\frac{u-u'+i}{u-u'-i}\right) = -\frac{1}{\pi}\frac{1}{(u-u')^2+1} && (2.9)\\ K_d(u,u') &= \frac{1}{2\pi}\frac{d}{du}\theta(u,u') \,. && (2.10) \end{aligned}$$

When the dressing phase is taken into account, the complete kernel is

$$\begin{aligned} \mathcal{K}_{\rm tot}(u,u') &= (1-\eta)K(u,u') + \eta K_{su(2)}(u,u') + K_d(u,u') \\ &\equiv \mathcal{K}(u,u') + \eta K_{su(2)}(u,u') \,. \end{aligned} \tag{2.11}$$

It will be sometimes convenient to work with the phase associated with the kernel $\mathcal{K}$:

$$\mathcal{K}(u,u') = \frac{1}{2\pi}\partial_u \phi(u,u') \,, \qquad \phi = (1-\eta)\varphi + \theta. \tag{2.12}$$

The $sl(2)$ sector is special, in the sense that the number of magnons, which is called also spin, $M = S$, is not bounded, even if the length is finite ($L = 2$ for the twist-two operator). The lowest state with S magnons can be characterized by the numbers

$$n_k = k + \mathrm{sign}(k) \,, \qquad k = -(S-1)/2, \dots, (S-1)/2 \,. \tag{2.13}$$

The corresponding integral equation is given, again in the absence of the dressing phase, by

$$\rho(u) = \frac{2}{s}\delta(u) + \frac{1}{\pi}\int_{-\infty}^{\infty} du' \, \frac{\rho(u')}{(u-u')^2+1} + 2\int_{-\infty}^{\infty} du' \, K(u,u') \, \rho(u') \,, \tag{2.14}$$

where the term proportional to $\delta(u)$ comes from the distribution of n_k. The limit $g = 0$ of this equation is singular, and it was solved in [39], its solution for large S being

$$\rho_0(u) = \frac{1}{\pi S} \ln \frac{1+\sqrt{1-4u^2/S^2}}{1-\sqrt{1-4u^2/S^2}} = \frac{2\ln S}{\pi S} + \mathcal{O}(1/S). \tag{2.15}$$

Eden and Staudacher [35] chose to separate the density $\rho(u)$ into the $g = 0$ part, $\rho_0(u)$, and a fluctuation $\sigma(u)$

$$\rho(u) = \rho_0(u) - 2g^2 \frac{4\ln S}{S}\, \sigma(u) + \mathcal{O}(1/S) = -2g^2 \frac{4\ln S}{S} \left(\sigma(u) - \frac{1}{4\pi g^2}\right) + \mathcal{O}(1/S) \,. \tag{2.16}$$

From (2.14), the equation satisfied by the density fluctuation $\sigma(u)$ is

$$\sigma(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} du' \, \frac{\sigma(u')}{(u-u')^2+1} + 2\int_{-\infty}^{\infty} du' \, K(u,u') \, \left(\sigma(u') - \frac{1}{4\pi g^2}\right) \,. \tag{2.17}$$

This is essentially the equation derived by Eden and Staudacher [35], with the inhomogeneous term written differently. This way of writing has the advantage to show that the separation of $\rho(u)$ into $\rho_0(u)$ and $\sigma(u)$ makes sense, at large g, if $K(u,u')$ is sufficiently well behaved at large u'. As we shall see, if $K(u,u') = \mathcal{O}(1)$ for large g and $|u'| > 2g$, the equation (2.17) will have only the trivial solution[4], since at leading order in $1/g$ one can write

$$\int_{-\infty}^{\infty} du' \, K(u,u') \, \left(\sigma(u') - \frac{1}{4\pi g^2}\right) = 0 \,. \tag{2.18}$$

In this case, the separation of the density into a free part and a perturbation is inconsistent. The Eden-Staudacher kernel is exactly of this type, and this might be the reason the strong coupling limit of the Eden-Staudacher equation is so pathological [22, 23]. The BES kernel, on the other hand, vanishes at the order $\mathcal{O}(1)$ for $|u'| > 2g$, and the strong coupling limit of the BES equation is well behaved.

[4] At least if the kernel is non-degenerate.

3 Elliptic parametrization of the kernels

3.1 The elliptic map

In the strong coupling regime, it is more convenient to rescale the variables u and $x^{\pm}$ in order to eliminate the overall factors of g. We will use the rescaled variables throughout the rest of the paper

$$u = \frac{u_{\rm old}}{2g}, \quad x^{\pm} = \frac{x^{\pm}_{\rm old}}{g}, \qquad \epsilon = \frac{1}{4g}\,. \tag{3.1}$$

The function

$$x(u) = u + \sqrt{u^2 - 1} \tag{3.2}$$

has a branch cut along the interval $[-1, 1]$. Since the Bethe equations involve the shifted variables $u \pm i\epsilon$, they contain two symmetric cuts, $[-1 - i\epsilon, 1 - i\epsilon]$ and $[-1 + i\epsilon, 1 + i\epsilon]$.

It can be advantageous to resolve the cuts by introducing a global parametrization. We eliminate the parameter u from

$$u_{\pm} \equiv u \pm i\epsilon = \tfrac{1}{2}(x^{\pm} + 1/x^{\pm}) \tag{3.3}$$

to obtain the following relation between x^+ and x^-:

$$\left(x^+ - x^-\right)\left(1 - \frac{1}{x^+ x^-}\right) = 4i\epsilon. \tag{3.4}$$

If we impose that the momentum p is real, we should also assume that x^+ and x^- are complex conjugate and u real. Then the condition (3.4) for $x = x^+$ and $\bar{x} = x^-$ define a contour $\mathcal{C}$ in the complex x-plane. The contour consists, for g real, of two connected components, $\mathcal{C}_\infty$ and $\mathcal{C}_0$, which are exchanged by the particle-antiparticle transformation $x \to 1/x$. We denote by $\mathcal{C}_\infty$ the component that contains the point $x = \infty$; the other component contains the point $x = 0$. In the limit $\epsilon \to 0$ the two contours $\mathcal{C}_\infty$ and $\mathcal{C}_0$ develop cusps where they touch (Fig.1). At this point the contour $\mathcal{C}$ is the union of the real line and the unit circle. A second singular value is $\epsilon = \pm i$. Here the two connected component join into one (Fig.2).

The equation for the contour $\mathcal{C}$ can be written conveniently in terms of the momentum p and a complementary parameter β:

$$x^{\pm} = e^{(\beta \pm ip)/2}\,. \tag{3.5}$$

In these variables the relation (3.4) reads

$$\sin\frac{p}{2}\sinh\frac{\beta}{2} = \epsilon \tag{3.6}$$

and the parameter u is exspressed as

$$u = \cos\frac{p}{2}\cosh\frac{\beta}{2}\,. \tag{3.7}$$

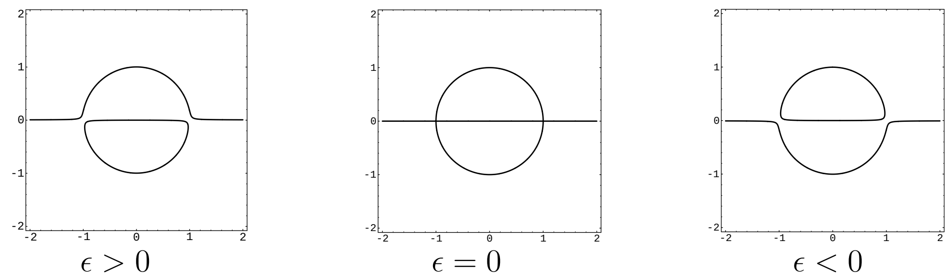


Fig.1 : The contour $\mathcal{C}$ in the x plane for three real values of ϵ close to 0.

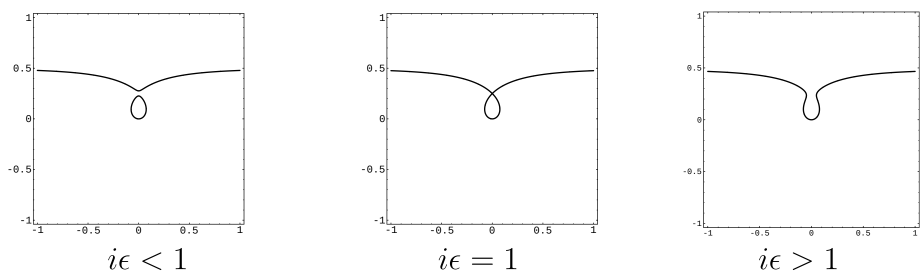


Fig. 2 : The contour $\mathcal{C}$ for three imaginary values of ϵ close to $-i$.

In the following, we consider only positive values of ϵ, and $p \in [0, 2\pi]$. This insures that u in (3.7) takes all the real values once and only once.

The relation (3.6) can resolved by an elliptic parametrization[5] with modulus k defined as

$$\epsilon = \frac{k'}{k} \qquad \text{or} \qquad k = \frac{1}{\sqrt{1+\epsilon^2}}\,. \tag{3.8}$$

Then β and p are parametrized by the Jacobi elliptic amplitude function:

$$p(s) = \pi - 2\,\mathrm{am}(\mathrm{K} - s, k), \quad \beta(s) = -i\pi - 2i\,\mathrm{am}(i\mathrm{K}' - s, k). \tag{3.9}$$

When the elliptic parameter s sweeps the interval $[0, 2\mathrm{K}]$, the momentum p increases from 0 to 2π. The symmetries of p and β are

$$p(-s) = -p(s), \quad p(s+2K) = p(s) + 2\pi \tag{3.10}$$

$$\beta(-s) = \beta(s) + 2\pi i, \quad \beta(2K - s) = \beta(s). \tag{3.11}$$

The functions we will work with are actually expressed in terms of Jacobi elliptic functions:

$$\begin{aligned} e^{\pm\beta/2} &= \frac{1 \pm \mathrm{dn}\, s}{k\,\mathrm{sn}\, s}, \quad e^{\pm ip/2} = \mathrm{cd}\, s \pm ik'\,\mathrm{sd}\, s \\ \partial_s p &= 2k'\mathrm{nd}\, s, \qquad \partial_s\beta = -2\,\mathrm{cs}\, s. \end{aligned} \tag{3.12}$$

[5]Our map is related to that of [9] by Gauss-Landen transformation: $k_{\mathrm{Janik}} = 2\sqrt{k'}/(1+k')$.

The contours $\mathcal{C}_\infty$ and $\mathcal{C}_0$ are parametrized respectively by $s \in I_\infty$ and $s \in I_0$, where

$$I_\infty = [0, 2\mathrm{K}], \quad I_0 = [2i\mathrm{K}', 2\mathrm{K} + 2i\mathrm{K}']. \tag{3.13}$$

The original variable u is parametrized as

$$u(s) = \frac{1}{k} \frac{\mathrm{cn}\, s}{\mathrm{sn}\, s\ \mathrm{dn}\, s}. \tag{3.14}$$

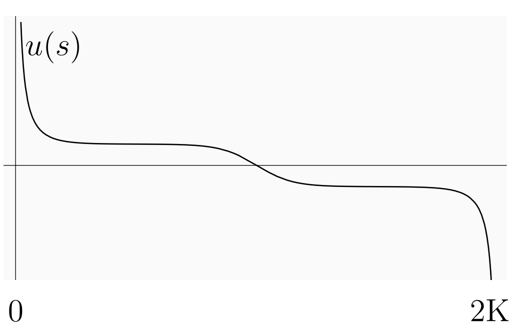
It obeys the symmetries $u(s) = -u(-s) = u(s + 2\mathrm{K}) = u(s + 2i\mathrm{K}')$, as well as

$$u(s)\, u(\mathrm{K} - s) \ = \ 1 + \epsilon^2\,. \tag{3.15}$$

The real axis in the u-plane is the image of the interval I_∞. The functions $x^\pm(s)$ are periodic in s with periods $2\mathrm{K}$ and $4i\mathrm{K}'$ and have the symmetries

$$x^+(s) = x^+(\mathrm{K} - i\mathrm{K}' - s) = -x^-(-s), \quad x^\pm(s + 2i\mathrm{K}') = \frac{1}{x^\pm(s)}. \tag{3.16}$$

The first symmetry preserves the contours $\mathcal{C}_\infty$ and $\mathcal{C}_0$, while the second symmetry exchanges them. Note that the factors $e^{\beta/2}$ and $e^{\pm ip/2}$ are themselves antiperiodic in $s \to s + 2\mathrm{K}$. This is a little complication, due to the fact that we are working with trigonometric functions of $p/2$, which are strictly speaking periodic with period 4π. However, in the physical variables, they always appear in combinations which are periodic with period 2π. As we already mentioned, we prefer to work with p in the interval $(0, 2\pi)$.



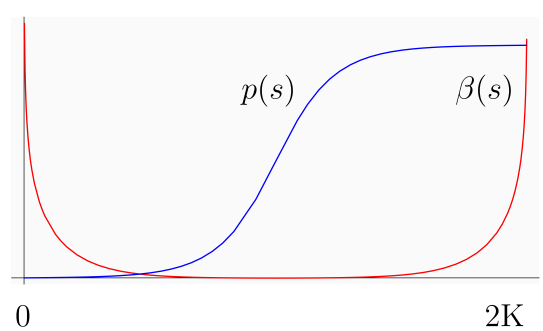


Fig. 3: Left: The function $u(s)$ for $\epsilon = 10^{-3}$ on the interval $[0, 2\mathrm{K}]$;
Right: The functions $p(s)$ (in blue) and $\beta(s)$ (in red) for $\epsilon = 10^{-3}$.

In the limit $\epsilon \to 0$, where $\mathrm{K} \to \infty$ and $\mathrm{K}' \to \pi/2$, the elliptic parametrization degenerates[6] and the variables x and u can be given by hyperbolic functions of s. As the plot

[6]Another point where the torus degenerates into a cylinder is $\epsilon = \pm i$, or $g^2 = -1/16$, *cf.* Fig.2.

of $u(s)$ in Fig. 3 suggests, the parametrization interval splits into two different pieces, corresponding to two different regimes. By periodicity[7], we can choose the parametrization interval as $[-\frac{1}{2}\mathrm{K}, \frac{3}{2}\mathrm{K}]$; then the first regime corresponds to the interval $[-\frac{1}{2}\mathrm{K}, \frac{1}{2}\mathrm{K}]$, which becomes the whole real axis when $\mathrm{K} \to \infty$. The second regime, corresponding to the interval $[\frac{1}{2}\mathrm{K}, \frac{3}{2}\mathrm{K}]$ can be mapped also to the interval $[-\frac{1}{2}\mathrm{K}, \frac{1}{2}\mathrm{K}]$ by the transformation

$$s \to \mathrm{K} - s. \tag{3.17}$$

The first interval corresponds to values $|u| > 1$, while the second corresponds to $|u| < 1$. We therefore introduce the suggestive notation

$$I^{>} = [-\tfrac{1}{2}\mathrm{K}, \tfrac{1}{2}\mathrm{K}] \quad \text{and} \quad I^{<} = [\tfrac{1}{2}\mathrm{K}, \tfrac{3}{2}\mathrm{K}] \tag{3.18}$$

and use the asymptotic formulas valid in the limit $\epsilon \to 0$ and for $s \in I^{>}$:

$$\begin{aligned} \mathrm{sn}\, s &= \tanh s, & \mathrm{sn}\,(\mathrm{K}-s) &= 1 \\ \mathrm{cn}\, s &= 1/\cosh s, & \mathrm{cn}\,(\mathrm{K}-s) &= k' \sinh s, \\ \mathrm{dn}\, s &= 1/\cosh s, & \mathrm{dn}\,(\mathrm{K}-s) &= k' \cosh s. \end{aligned} \tag{3.19}$$

The next terms in the expansion in ϵ are given in Appendix A.

In the hyperbolic limit, the expression (3.14) for u becomes, in the two different regimes,

$$u^{>}(s) = u(s) \simeq \coth s\ , \qquad u^{<}(s) = u(\mathrm{K}-s) \simeq \tanh s\ . \tag{3.20}$$

The points $s = \pm \mathrm{K}/2$, which are the fixed points of the transformation $s \to \mathrm{K} - s$, correspond to the points $u = \pm\sqrt{1+\epsilon^2}$. In the strong coupling limit, $\epsilon \to 0$, these are the points $u = \pm 1$, and they correspond to meeting points $s \to \pm\infty$ of the two parametrizations (3.20). These points will be very important, and we need a better parametrization for them. In the strong coupling limit the vicinity of these points is parametrized by (Appendix A)

$$u(\pm\tfrac{1}{2}\mathrm{K} + s) = \pm 1 - \epsilon \sinh 2s + \mathcal{O}(\epsilon^2). \tag{3.21}$$

The expressions for the momentum p and its counterpart β in this regimes are

$$\sinh \beta/2 = \sqrt{\epsilon}\, e^{\mp s} \qquad \sin p/2 = \sqrt{\epsilon}\, e^{\pm s}\ , \tag{3.22}$$

where the upper sign correspond to momenta close to 0 and the lower sign to momenta close to 2π. A similar parametrization was used in [26].

3.2 Strong coupling limit of the $su(1|1)$ kernel

We saw that in the strong coupling limit the integration interval splits naturally into two domains, $|u| < 1$ and $|u| > 1$, which are the images of the intervals $I^{<}$ and $I^{>}$ in the

[7] As already explained, the variables u and x are periodic with period 2K, and we can safely do the shift $s \to s - 2\mathrm{K}$, but for the variables p, β we have to remember that the point $-\frac{1}{2}\mathrm{K}$ was originally $\frac{3}{2}\mathrm{K}$.

s parametrization. As it is clear from Fig. 1, in the first domain the complex variable $x = x^+$ becomes asymptotically unimodular, while in the second domain it becomes asymptotically real. For $\epsilon \to 0$ the relations (3.6) and (3.7) give

$$\beta \to 0\,, \quad u = \cos p/2 \qquad \text{if} \quad |u| < 1\,, \tag{3.23}$$
$$p \to 0\,, \quad u = \cosh \beta/2 \qquad \text{if} \quad |u| > 1\,. \tag{3.24}$$

Passing to the elliptic parameter s we write the integral equation (2.6) as

$$\rho(s) = \frac{1}{2\pi} p'(s) + \int_{I_\infty} ds_1\, K(s,s_1)\, \rho(s_1)\,, \tag{3.25}$$

where

$$\rho(s) = |u'(s)|\, \rho(u), \qquad K(s,s_1) = |u'(s)|\, K(u,u_1). \tag{3.26}$$

The first term changes sign because the derivative $u'(s)$ is negative. The kernel is equal to the derivative

$$\begin{aligned} K(s,s_1) &= -\frac{1}{2\pi i}\frac{d}{ds}\left[\ln\left(1 - e^{-\frac{1}{2}(\beta+\beta_1)} e^{-i\frac{1}{2}(p-p_1)}\right) - \ln\left(1 - e^{-\frac{1}{2}(\beta+\beta_1)} e^{i\frac{1}{2}(p-p_1)}\right)\right] \\ &= \frac{1}{4\pi} p'(s) - \frac{1}{4\pi}\frac{p'(s)\sinh\frac{1}{2}(\beta+\beta_1) - \beta'(s)\sin\frac{1}{2}(p-p_1)}{\cosh\frac{1}{2}(\beta+\beta_1) - \cos\frac{1}{2}(p-p_1)}. \end{aligned} \tag{3.27}$$

Let us mention here that the piece $p'(s)/4\pi$ has a physical meaning: it corresponds to a change of the effective length of the chain. Every magnon increases the effective length of the chain by 1/2. This is particularly clear on the discrete equations, where, at strong coupling, the Bethe equations become, as it was noticed in [40]

$$e^{ip_k(L+M/2)} = 1 \tag{3.28}$$

under the condition that all p_k are finite.

It is now easy to take the strong coupling limit using the asymptotical expressions for $p(s)$ and $\beta(s)$, which can be found in Appendix A. The asymptotic form of the kernel is given by four different analytic expressions depending on whether its arguments are in the interval $I^>$ or $I^<$. After applying the redefinition (3.17) to the arguments that belong to the second interval, we can write the result as a 2×2 matrix kernel

$$\begin{pmatrix} K^{>>}(s,s_1) & K^{><}(s,s_1) \\ K^{<>}(s,s_1) & K^{<<}(s,s_1) \end{pmatrix} := \begin{pmatrix} K(s,s_1) & K(s,\mathrm{K}-s_1) \\ K(\mathrm{K}-s,s_1) & K(\mathrm{K}-s,\mathrm{K}-s_1) \end{pmatrix} \tag{3.29}$$

defined in the interval $I^>$, which extends to $[-\infty,\infty]$ when $\epsilon \to 0$. Similarly the density splits into two components,

$$\begin{pmatrix} \rho^>(s) \\ \rho^<(s) \end{pmatrix} = \begin{pmatrix} \rho(s) \\ \rho(\mathrm{K}-s) \end{pmatrix} \tag{3.30}$$

satisfying, in the limit $\epsilon \to 0$, the normalization condition[8]

$$\int_{-\infty}^{\infty} ds \rho^< + \int_{-\infty}^{\infty} ds \rho^> = 1. \tag{3.31}$$

[8]This is true only for the kernel without the dressing phase, at the leading order.

Eq. (3.25) now takes the form

$$\rho^{>}(s) = \int_{-\infty}^{\infty} ds_1 K^{>>}(s,s_1)\,\rho^{>}(s_1) + \int_{-\infty}^{\infty} ds_1 K^{><}(s,s_1)\,\rho^{<}(s_1)\,, \tag{3.32}$$

$$\rho^{<}(s) = \frac{p'(s)}{2\pi} + \int_{-\infty}^{\infty} ds_1 K^{<>}(s,s_1)\,\rho^{>}(s_1) + \int_{-\infty}^{\infty} ds_1 K^{<<}(s,s_1)\,\rho^{<}(s_1). \tag{3.33}$$

The matrix elements of the kernel (3.29), evaluated in Appendix A, are

$$\begin{aligned} K^{<<}(s,s') &= \frac{1}{2\pi}\frac{1}{\cosh s} - \delta(s-s')\,, \\ K^{<>}(s,s') &= \frac{1}{2\pi}\frac{1}{\cosh s} - \frac{1}{2\pi}\frac{1}{\cosh(s-s')}\,, \\ K^{><}(s,s') &= \frac{1}{2\pi}\frac{1}{\cosh(s-s')}\,, \\ K^{>>}(s,s'') &= 0\,. \end{aligned} \tag{3.34}$$

Note that in the hyperbolic limit all four integration kernels depend on the difference of the arguments. This is a considerable simplification, the equation can be now solved using the Fourier transform.

In the third regime, which corresponds to the points around $u = \pm 1$ the momenta are parametrized by (3.22). This regime is very important for evaluation of the density with the BHL/BES kernel. Here, the appropriate parametrization is given by (3.21) and (3.22). The only component of the kernel that survives at the leading order is when either $u, u' \simeq 1$ or $u, u' \simeq -1$

$$\begin{aligned} &K^{++}(s,s') = K^{--}(-s,-s') \\ &= -\frac{1}{4\pi}\frac{\sinh(s-s')+1}{\cosh(s-s')\cosh(s+s') - \sinh(s+s')} + \mathcal{O}(\sqrt{\epsilon})\,. \end{aligned} \tag{3.35}$$

If only one of the arguments is in the intermediate region, the kernel is of order $\sqrt{\epsilon}$.

One might argue that there are many intermediate regimes, where p scales like an arbitrary power of ϵ,

$$p \sim \epsilon^{1-\gamma}\,, \qquad \beta \sim \epsilon^{\gamma} \quad \gamma \in (0,1)\,. \tag{3.36}$$

However, if $\gamma \neq 1/2$, one of the variables p and β dominates the other, and these regimes are properly taken into account in the regions $\gtrless$. Only for $\gamma = 1/2$ the two contributions balance each other and this is why we have to treat this case separately.

4 Solving the integral equations in the strong coupling limit without the dressing kernel

4.1 The $su(2)$ case

The $su(2)$ kernel is the only one which is of difference form, therefore it can be exactly solved for any value of g [31, 32]. The energy of the antiferromagnetic state is identical to the energy of the ground state for the Hubbard model at half filling [42]. Here, we would like to comment on the strong coupling limit of this solution. The integral equation is particularly simple, since the kernel becomes simply the delta function

$$K_{su(2)}(u,u') = -\lim_{\epsilon\to 0}\frac{1}{\pi}\frac{2\epsilon}{(u-u')^2+4\epsilon^2} = -\delta(u-u')\,. \tag{4.1}$$

The integral equation reduces then to

$$2\rho(u) = -\frac{1}{2\pi}\frac{dp}{du}\,, \qquad \text{or} \qquad \rho(p) = \frac{1}{4\pi}\,. \tag{4.2}$$

The solution corresponds to $L/2$ roots distributed uniformly between $p=0$ and 2π. In this case, the density is normalized to $1/2$, because the maximum number of magnons is $L/2$. All the solutions correspond to the interval $|u|<1$ and the energy is

$$E_{su(2)} = 4gL\int_0^{2\pi} dp\,|\sin p/2|\,\rho(p) = \frac{4gL}{\pi}\,. \tag{4.3}$$

Physically, the excitations are the (usual) magnons. The solution obtained above shows that, in the strong coupling limit, the magnons are *free*, except for the statistical repulsion (4.1). The situation is very much similar to the Haldane-Shastry model [43, 44], where the magnons are also free up to the statistical interaction, translated into the rule that two magnons cannot occupy succesive momenta. Let us remind that the scattering phase of the for particles with purely statistical interaction is [45]

$$\varphi(p,p') = (\lambda-1)\,\pi\,\mathrm{sign}(p-p')\,, \tag{4.4}$$

which is consistent with (4.1) with the statistical parameter of the magnons $\lambda=2$. The only difference with the Haldane-Shastry spin chain is that here the magnons have the dispersion relation

$$E(p) = 4g|\sin p/2|\,, \tag{4.5}$$

which is typical for a finite-difference hamiltonian, while the Haldane-Shastry magnons have the dispersion relation

$$E_{HS}(p) = p(2\pi - p)\,. \tag{4.6}$$

A natural candidate for a model with purely statistical interaction and with trigonometric dispersion relation is the Ruijenaars-Schneider model [46]. This hints about the existence of a spin model which would describe the spin sector of the half-filled Hubbard model for any value of g and which might be a multi-spin deformation of the Inozemtsev model [47, 48].

4.2 The $su(1|1)$ case

The solution of the $su(1|1)$ sector is considerably more involved, but its leading order still can be obtained in closed form. Since now the roots of the Bethe equations occur both in the regions $|u|<1$ and $|u|>1$, the integral equation splits up into two coupled equations for the densities in the two regions.

$$2\rho^{<}(s) = \frac{3}{2\pi}\frac{1}{\cosh s} - \frac{1}{2\pi}\int_{-\infty}^{\infty} ds'\,\frac{\rho^{>}(s')}{\cosh(s-s')}\,, \tag{4.7}$$

$$\rho^{>}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds'\,\frac{\rho^{<}(s')}{\cosh(s-s')}\,. \tag{4.8}$$

Substituting the second equation into the first we obtain a new equation where the kernel is of difference form

$$\rho^{<}(s) = \frac{3}{4\pi}\frac{1}{\cosh s} - \frac{1}{4\pi^2}\int_{-\infty}^{\infty} ds'\,\frac{s-s'}{\sinh(s-s')}\,\rho^{<}(s')\,, \tag{4.9}$$

and therefore is solved by Fourier transform. In Fourier space the equation reads[9]

$$\rho^{<}(t) = \frac{3}{4\pi}\frac{\pi}{\cosh(\pi t/2)} - \frac{1}{4\pi^2}\frac{\pi^2}{2\cosh^2(\pi t/2)}\rho^{<}(t)\,, \tag{4.10}$$

with the solution

$$\rho^{<}(t) = \frac{6\cosh(\pi t/2)}{1+8\cosh^2(\pi t/2)}. \tag{4.11}$$

Transforming back to the s-variable, we get

$$\rho^{<}(s) = \frac{1}{\sqrt{2}\pi}\frac{\cos 2as/\pi}{\cosh s}\,, \tag{4.12}$$

$$a = \ln\sqrt{2} \simeq .34657\,. \tag{4.13}$$

For the density of roots $\rho(u)$ we obtain in the interval $-1<u<1$

$$\rho^{<}(u) = \frac{\rho^{<}(s)}{|u'(s)|} = \frac{1}{\sqrt{2}\pi}\frac{1}{\sqrt{1-u^2}}\cos\left(\frac{a}{\pi}\ln\frac{1+u}{1-u}\right)\,. \tag{4.14}$$

To obtain the function $\rho^{>}(u)$, we make the change of variable $u=\coth s$, which is appropriate for $u>1$, so that we get

$$\rho^{>}(s) = \frac{\sinh s}{2\sqrt{2}\pi^2}\int_{-\infty}^{\infty}\frac{ds'}{\cosh s'}\,\frac{\cos 2as'/\pi}{\cosh(s-s')} = \frac{1}{\pi}\sin\frac{2as}{\pi}\,, \tag{4.15}$$

or, in the initial variables,

$$\rho^{>}(u) = \frac{1}{\pi}\frac{1}{\sqrt{u^2-1}}\sin\left(\frac{a}{\pi}\ln\frac{u+1}{u-1}\right)\,. \tag{4.16}$$

[9] In order to keep the notation simple, we are going to use the same symbol for the Fourier transforms $\rho^{<}(t)$, $\rho^{>}(t)$ of the densities.

The result shows a singularity at the points $u = \pm 1$; moreover, it can take negative values; this happens at values of $1 - u \simeq 10^{-5}$ and is presumably due to the roots around $u = \pm 1$ which were not taken properly into account. Except for these oscillations, the result superposes beautifully with the solution obtained by numerical integration of the Fredholm equation[10].

The total normalization of the density is insured, since $\int_< du\ \rho_<(u) = 2/3$ and $\int_> du\ \rho_>(u) = 1/3$. So two third of the roots of the Bethe equations fall in the region $|u| < 1$ and one third in the intervals $|u| > 1$. This phenomenon is also present in the numerical solution of [33]. Finally, the energy of the state is given by

$$E_{su(1|1)} = 4gL \int_{-1}^{1} du\ \rho_<(u)\ \sqrt{1-u^2} \tag{4.17}$$

$$= \frac{2\sqrt{2}gL}{\pi} \int_{-\infty}^{\infty} ds\ \frac{\cos 2as/\pi}{\cosh^2 s} = \frac{16a}{\pi} gL\ . \tag{4.18}$$

This result is consistent with that of [33], $c_L = 2\ln(2)/\pi^2 \simeq .14046$.

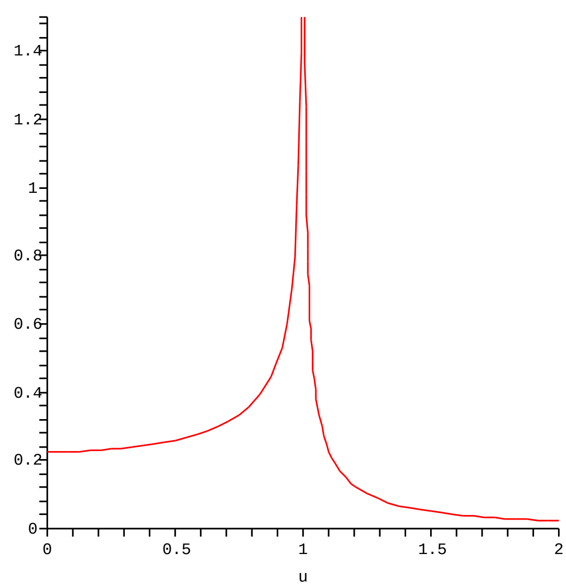


Fig. 4: Density profile predicted by the analytic solution (4.14), (4.16). Only the $u > 0$ part is shown. The oscillations do not show yet at the chosen scale.

4.3 The $sl(2)$ case

The dimension of the twist two operator was originally supposed to be determined by the solution of Eden-Staudacher equation [35]

$$\sigma(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} du'\ \frac{2\epsilon}{(u-u')^2 + 4\epsilon^2}\ \sigma(u') + 2 \int du'\ K(u,u')\ \left(\sigma(u') - \frac{1}{2\pi g}\right)\ . \tag{4.19}$$

[10] We thank M. Staudacher for supplying us the solution obtained by numerical integration.

The anomalous dimension is simply given by the normalization of the density fluctuation $\sigma(u)$ [49]

$$f(g) = 16g^2 \int_{-\infty}^{\infty} du \, \sigma(u) \ , \tag{4.20}$$

or alternatively by the formula [35]

$$f(g) = 8g^2 \left(1 - 2g \int_{-\infty}^{\infty} du \, \sigma(u) \left(\frac{i}{x^+(u)} - \frac{i}{x^-(u)}\right)\right) \ . \tag{4.21}$$

As explained in section 2, in the limit $\epsilon = 1/4g \to 0$ the equation (4.19) becomes

$$\int du' \; K(u,u') \; \left(\sigma(u') - \frac{1}{2\pi g}\right) = 0 \ . \tag{4.22}$$

Obviously, this equation has as solution[11]

$$\sigma(u) = \frac{1}{2\pi g} \tag{4.23}$$

on the whole real axis. This solution, first obtained in [35], seems to be correct in the interval $|u| < 1$, since it insures the vanishing of the $\mathcal{O}(g^2)$ term in $f(g)$

$$f(g) = 8g^2 \left(1 - 4g \int_{-1}^{1} du \, \sigma^<(u)\sqrt{1-u^2} + \mathcal{O}(1/g)\right) = \mathcal{O}(g) \ . \tag{4.24}$$

In the interval $|u| > 1$, although it has the right scaling in g, the solution (4.23) is not integrable and it gives formally an infinite value for the anomalous dimension of the twist-two operator. As mentioned in the section 2, the density (4.23) exactly compensate the leading order in S in $\rho_0(u)$ *everywhere* in u

$$\rho_0(u) = \frac{4g}{\pi} \frac{\ln S}{S} + \dots \ . \tag{4.25}$$

Unfortunately, as we show below, (4.23) is the unique solution to the ES equation at strong coupling. We can write the equations as

$$\begin{aligned} \frac{1}{2\pi} \int ds' \, \frac{\sigma^<(s')}{\cosh(s-s')} \quad &= \quad \frac{1}{8\pi g} \frac{1}{\cosh^2(s/2)} \ , \\ \frac{1}{2\pi} \int ds' \, \frac{\sigma^>(s')}{\cosh(s-s')} + \sigma^<(s) \quad &= \quad \frac{1}{4\pi g} \frac{1}{\cosh^2(s)} + \frac{1}{2\pi} \frac{A}{\cosh(s)} \ , \end{aligned} \tag{4.26}$$

where $A = \int ds \; (\sigma^<(s) + \sigma^>(s))$ is the normalization of the density. In Fourier transformed form, the equations become

$$\begin{aligned} \frac{\sigma^<(t)}{2\cosh \pi t/2} \quad &= \quad \frac{t}{2g \sinh \pi t} \ , \\ \frac{\sigma^>(t)}{2\cosh \pi t/2} + \sigma^<(t) \quad &= \quad \frac{t}{4g \sinh \pi t/2} + \frac{A}{2\cosh \pi t/2} \ . \end{aligned} \tag{4.27}$$

[11]Here we use the variable u rescaled by $2g$, whence the extra factors of $2g$ in the density.

The first line is readily solved as

$$\sigma^{<}(t)=\frac{1}{2g\sinh\pi t/2}\,,\quad \text{or}\quad \sigma^{<}(s)=\frac{1}{2\pi g}\frac{1}{\cosh^2 s}\,,\quad \text{or}\quad \sigma^{<}(u)=\frac{1}{2\pi g}\,,\tag{4.28}$$

while the solution for the second is

$$\begin{aligned}\sigma^{>}(t)=A-\frac{t}{2g\pi\tanh\pi t/2}\,,\qquad &\text{or}\qquad \sigma^{>}(s)=A\,\delta(s)+\frac{1}{2\pi g}\frac{1}{\sinh^2 s}\,,\\ \text{or}\qquad \sigma^{>}(u)\;&=\;-A\,\delta(u-\infty)+\frac{1}{2\pi g}\,.\end{aligned}\tag{4.29}$$

The normalization A is not fixed by the equation.

All other approaches tried to compute the strong coupling limit from the Eden-Staudacher ran into some pathology as well: the result obtained by Kotikov and Lipatov [22] oscillates strongly, the result obtained numerically in [23] may not converge to a straight line in g as expected[12] and the numerical solution for $\sigma(u)$ in [35] does not seem to converge to a value with definite normalization when $g\to\infty$. We believe that all these pathologies are related to the separation of the density $\rho(u)$ into $\rho_0(u)$ and $\sigma(u)$. Rather surprisingly, this pathology disappear when the dressing kernel is taken into account, as we will show in the next section.

5 The dressing kernel

5.1 Integral representation

Beisert, Eden and Staudacher proposed an expression for the dressing factor in the Bethe ansatz equations. This dressing factor translates into a correction to the kernel in the integral equation. Their formula is given in Fourier transformed form; for our purposes an expression in terms of variables u is more suited. Let us start with the Fourier transform of the $su(1|1)$ kernel[13]

$$\hat{K}(t,t')=-\pi(1-\ \mathrm{sign}\,tt')|t|e^{-(|t|+|t'|)\epsilon}\sum_{n>0}n\,\frac{J_n(|t|)J_n(|t'|)}{|tt'|}\,.$$

which is related to the kernel of [13] by

$$\hat{K}(t,t')=-\frac{\pi}{2}(1-\ \mathrm{sign}\,tt')|t|e^{-(|t|+|t'|)\epsilon}\hat{K}_m(|t|,|t'|)\tag{5.1}$$

and can be broken into a symmetric and antisymmetric part:

$$\begin{aligned}\hat{K}(t,t')\;&=\;\hat{K}_+(t,t')+\hat{K}_-(t,t')\\ &=\;-\frac{\pi}{2}(1-\ \mathrm{sign}\,tt')|t|e^{-(|t|+|t'|)\epsilon}(K_0(|t|,|t'|)+K_1(|t|,|t'|))\,.\end{aligned}$$

[12]We thank the authors of [23] for checking the behavior of their ES solution.

[13]Our definition of the kernel differs by a minus sign from the one of [35].

In the ES and BES equation, the kernel is written in the Fourier variables defined only for positive values of t and t'. This is made possible since the densities $\rho(u)$ and $\sigma(u)$ are symmetric under $u \to -u$, and therefore under $t \to -t$ for the states under consideration. For generic states, this symmetry is not present, and this is why it is desirable to have non symmetric kernel at hand. The extension of $K_d(t,t')$ to negative values of t and t' is ambiguous, and undoing the Fourier transform in the "magic" formula of [13] will be ambiguous, too .

The authors of [24] gave a prescription to to define $K_d(u,u')$ by symmetrizing all the kernels on the second variable $K(u,u') \to \frac{1}{2}(K(u,u') + K(u,-u'))$. We try to avoid symmetrizing the kernels from the beginning, and give here a different prescription, using instead that the full BES kernel is antisymmetric with respect to $u_\pm \to u_\mp$ on each variable separately. We remind that the transformations $u \to -u$ and $u_\pm \to u_\mp$ translate in the following way on the variable p

$$u \to -u \qquad \Leftrightarrow \qquad p \to 2\pi - p \tag{5.2}$$

$$u_\pm \to u_\mp \qquad \Leftrightarrow \qquad p \to -p \tag{5.3}$$

while the variable β stays unchanged. Of course, these two transformations act differently on the Fourier transform.

Here, for the sake of simplicity, we prefer to work with unsymmetrized kernels and impose the antisymmetry under (5.3) at the end of the computation. We define, for $t, t' > 0$

$$\begin{aligned}
\hat{K}_d(t,-t') &\equiv -4\pi t e^{-(t+t')\epsilon}\hat{K}_c(t,t') \\
&= -4\pi t e^{-(t+t')\epsilon}\int_0^\infty dt''\hat{K}_1(t,t'')\frac{t''}{e^{2\epsilon t''}-1}\hat{K}_0(t'',t') \\
&= -\frac{4}{\pi}\int_{-\infty}^\infty dt''\hat{K}_-(t,-t'')\frac{e^{2\epsilon|t''|}}{e^{2\epsilon|t''|}-1}\hat{K}_+(t'',-t')\ .
\end{aligned}$$

We conclude that we can define the dressing kernel for any values of t, t' as a convolution

$$\hat{K}_d(t,t') = -\frac{4}{\pi}\int_{-\infty}^\infty dt''\hat{K}_-(t,t'')\frac{e^{2\epsilon|t''|}}{e^{2\epsilon|t''|}-1}\hat{K}_+(-t'',t')\ , \tag{5.4}$$

where $\hat{K}_d(t,t')$ is also proportional to $1 - \text{sign}\, tt'$. After anti-symmetrization under (5.3) this will not be true anymore. Going back to the variables u we obtain

$$K_d(u,u') = -8\int_{-\infty}^\infty dv\ K_-(u,v)K_+(v,u') - \frac{4}{\epsilon}\int_{-\infty}^\infty dv\ dv' K_-(u,v)h(v-v')\tilde{K}_+(v',u') \tag{5.5}$$

where

$$h(u) = \frac{2\epsilon}{2\pi}\int_{-\infty}^\infty dt\ \frac{|t|e^{itu}}{e^{2\epsilon|t|}-1} = \frac{2\epsilon}{2\pi}\left(\frac{1}{u^2} - \frac{(\pi/2\epsilon)^2}{\sinh^2(\pi u/2\epsilon)}\right) \tag{5.6}$$

and $\tilde{K}_+(u,u')$ is the Fourier transform of $\hat{K}_+(t,t')/|t|$. The function $h(u)$ becomes $\delta(u)$ as ϵ approaches 0. Since the kernels involved in the convolution in (5.5) are of order ϵ^0, the dressing kernel will be of order ϵ^{-1}. This formula is very close to the one in

[24], except that we traded the difficulty involved in evaluating the function $h(u)$ for the difficulty in evaluating the kernel $\tilde{K}_+(u,v)$.

Let us now write down the expressions of $K_+(u,u')$ and $K_-(u,u')$, which contain the odd, respectively even powers in the expansion of the logarithm

$$\begin{aligned} K_-(u,v) &= \frac{1}{4\pi i}\partial_u \ln \frac{(1-X_{+-})(1+X_{+-})}{(1-X_{-+})(1+X_{-+})}\,, \qquad (5.7)\\ K_+(u,v) &= \frac{1}{4\pi i}\partial_u \ln \frac{(1-X_{+-})(1+X_{-+})}{(1+X_{+-})(1-X_{-+})}, \\ \tilde{K}_+(u,v) &= -\frac{1}{4\pi} \ln \frac{(1-X_{+-})(1-X_{-+})}{(1+X_{+-})(1+X_{-+})} \qquad (5.8)\end{aligned}$$

with

$$X_{+-} = \frac{1}{x^+(u)x^-(v)}\,, \qquad X_{-+} = \frac{1}{x^-(u)x^+(v)}\,. \qquad (5.9)$$

A strategy to compute the expansion of K_d in the strong coupling limit is to compute the limit of the auxiliary kernels (5.8), and then perform the integrals, as it was done in [24]. This procedure becomes rapidly quite involved, since different pieces of the kernel contribute for different orders in ϵ.

In the following, we give yet a different representation for the dressing kernel, which may be more useful for the strong coupling expansion. The equation (5.4) can be written as

$$\hat{K}_d(t,t') = -\frac{4}{\pi}\sum_{n\geq 0}\int_{-\infty}^{\infty} dt''\hat{K}_-(t,t'')e^{-2n\epsilon|t''|}\hat{K}_+(-t'',t')\,. \qquad (5.10)$$

This formula can be easily Fourier transformed back such that we get

$$K_d(u,u') = -8\sum_{n\geq 1}\int_{-\infty}^{\infty} dv\ K_-^{1,n}(u,v)\ K_+^{n,1}(v,u') \qquad (5.11)$$

with

$$\begin{aligned} K_{\mp}^{m,n}(u,v) &= -\frac{1}{2\pi i}\partial_u \sum_{l>0;\ {\text{even} \atop \text{odd}}} \frac{1}{l}\Big((x^{(+m)}(u)x^{(-n)}(v))^{-l} - (x^{(-m)}(u)x^{(+n)}(v))^{-l}\Big)\,, \\ x^{(\pm n)}(u) &= u \pm in\epsilon + \sqrt{(u\pm in\epsilon)^2-1}\,. \qquad (5.12)\end{aligned}$$

The expression (5.11) for the dressing kernel involves a sum of integrals along an infinite set of contours $\mathcal{C}^\infty_{n\epsilon}$, $n=1,2,3,\dots$, depicted in Fig. 5, defined in the same way as the contour $\mathcal{C}^\infty$, but with ϵ replaced by $n\epsilon$:

$$\mathcal{C}^\infty_{n\epsilon} = \{x\in\mathbb{C}\ \big|\ (x-\bar{x})\,(1-1/x\bar{x}) = 4in\epsilon, \quad \bar{x}x>1\}\,. \qquad (5.13)$$

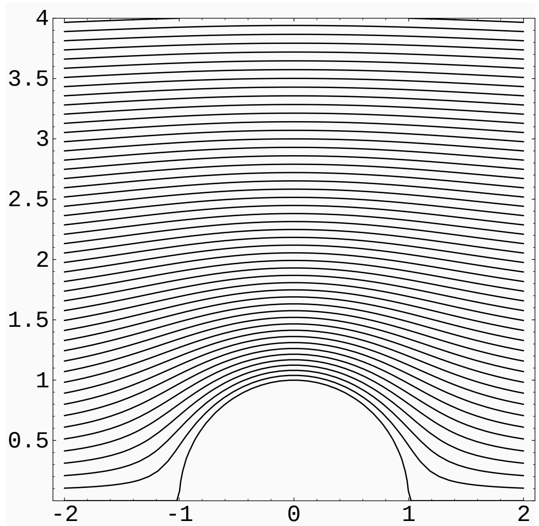


Fig. 5: The sequence of contours in the definition of the dressing kernel for $\epsilon = 0.04$.

In the strong coupling limit, the expression (5.11) becomes tractable, since we can evaluate the sum using the Euler-Maclaurin formula

$$\sum_{n=1}^{\infty} f(n\epsilon) = \frac{1}{\epsilon}\int_0^{\infty} f(z)\,dz - \frac{1}{2}f(0) - \sum_{k=1}^{\infty} \epsilon^{2k-1}\frac{B_{2k}}{(2k)!}f^{(2k-1)}(0). \tag{5.14}$$

We thus extend $n\epsilon$ to the continuous variable

$$n\epsilon = z \tag{5.15}$$

and define the contour $\mathcal{C}_z^{\infty}$ by

$$\mathcal{C}_z^{\infty}: \qquad \sin\frac{p}{2}\sinh\frac{\beta}{2} = z, \quad \beta > 0. \tag{5.16}$$

The integrals over v and the variable z become a double integral over the domain $\mathcal{D}$ in the complex x-plane spanned by the contours $\mathcal{C}_z^{\infty}$. The domain in question is the the upper half plane minus the unit semi-disk. In terms of the complexified momentum $\beta + ip = 2\ln x$, the domain $\mathcal{D}$ is the semi-infinite cylinder $\mathcal{D} = \{0 \le p < 2\pi, \beta > 0\}$.

The integral can be done without specifying a particular parametrization of the contour $\mathcal{C}_z^{\infty}$ if we express the integrand as a wedge product of differential forms. We first write the basic kernel as a 1-form:

$$K(x, x_1) = -\frac{dp}{4\pi} + \frac{1}{4\pi}\frac{\sinh\frac{\beta+\beta_1}{2}\,dp - \sin\frac{p-p_1}{2}d\beta}{\cosh\frac{\beta+\beta_1}{2} - \cos\frac{p-p_1}{2}}. \tag{5.17}$$

Then the "even" kernel K_+ is a one-form in the basis of dp and $d\beta$,

$$K_+(x, x_2) \;=\; \frac{1}{2\pi}\frac{\sinh\frac{\beta+\beta_2}{2}\cos\frac{p-p_2}{2}\,dp - \sin\frac{p-p_2}{2}\cosh\frac{\beta+\beta_2}{2}\,d\beta}{\cosh(\beta+\beta_2) - \cos(p-p_2)}\,, \tag{5.18}$$

while the "odd" kernel K_- is a zero-form,

$$K_-(x_1,x) \;=\; -\frac{1}{4\pi}dp_1 + \frac{1}{4\pi}\frac{dp_1\sinh(\beta+\beta_1) - d\beta_1\sin(p_1-p)}{\cosh(\beta+\beta_1)-\cos(p_1-p)}\,. \tag{5.19}$$

Also, for the differential dz we have from (5.16)

$$dz = \tfrac{1}{2}\sinh(\beta/2)\cos(p/2)\,dp + \tfrac{1}{2}\sin(p/2)\cosh(\beta/2)\,d\beta\,. \tag{5.20}$$

Then the leading term in the strong coupling limit $\epsilon\to 0$ is given by

$$K_d^0(x_1,x_2) = -\frac{8}{\epsilon}\int_0^\infty dz \int_{x\in\mathcal{C}_z} K_-\,K_+ \;\equiv\; -\frac{8}{\epsilon}\int_{\mathcal{D}} K_-\,dz\wedge K_+. \tag{5.21}$$

The evaluation of this integral (some details are given in Appendix B) leads to the following expression for K_d^0 :

$$\begin{aligned} K_d^0(x_1,x_2) &= \frac{1}{4\pi\epsilon}\,\partial_u\left[-\tilde\chi(x_1^-,x_2^+) + \tilde\chi(x_2^+,x_1^-) + \tilde\chi(x_1^-,-x_2^+) - \tilde\chi(-x_2^+,x_1^-) + \text{c.c.}\right], \\ \tilde\chi(x,y) &= -(x+\frac{1}{x})\log\left(y-\frac{1}{x}\right). \end{aligned} \tag{5.22}$$

This expression must be antisymmetrized with respect to $x_2^\pm \to x_2^\mp$ (or, equivalently, $p_2\to -p_2$, $\beta_2\to\beta_2$). After that the result coincides[14] with the AFS kernel $K^{AFS}(u,v)$, symmetrized with respect to $v\to -v$.

5.2 The strong coupling limit from the AFS kernel

Although we obtained an integral representation for the dressing kernel, we find it more straightforward to compute the first order(s) of the dressing kernel by using its standard strong coupling representation [12, 13]. The strong coupling expansion of the dressing phase can be cast in the following form

$$\begin{aligned} \theta_{12} &= \frac{1}{2\epsilon}\left[\chi_{12}^{--} - \chi_{12}^{+-} - \chi_{12}^{-+} + \chi_{12}^{++} - (1\leftrightarrow 2)\right] \\ \chi_{12}^{rs} &= \chi\left(x_1^r, x_2^s\right)\,, \quad r,s=\pm\,, \end{aligned} \tag{5.23}$$

where the function χ can be expanded in powers of the inverse coupling constant

$$\chi = \sum_{n\geq 0}\chi_n (2\epsilon)^n\,. \tag{5.24}$$

To compute the leading term in the anomalous dimension, the AFS term [10] is sufficient,

$$\chi_0(x,y) = -\frac{xy-1}{y}\ln\frac{xy-1}{xy}\,, \tag{5.25}$$

[14] It is useful to notice that we can substitute χ_0 with $\tilde\chi$ to calculate AFS term in(5.23).

while for the next correction we will need the Hernández-Lopez [11, 50] term

$$
\begin{aligned}
\chi_1(x,y) &= \frac{1}{\pi}\Big[\ln\frac{y-1}{y+1}\ln\frac{x-1/y}{x-y} \\
&+\mathrm{Li}_2\frac{\sqrt{y}-1/\sqrt{y}}{\sqrt{y}-\sqrt{x}}-\mathrm{Li}_2\frac{\sqrt{y}+1/\sqrt{y}}{\sqrt{y}-\sqrt{x}}+\mathrm{Li}_2\,\frac{\sqrt{y}-1/\sqrt{y}}{\sqrt{y}+\sqrt{x}}-\mathrm{Li}_2\frac{\sqrt{y}+1/\sqrt{y}}{\sqrt{y}+\sqrt{x}}\Big]\,.
\end{aligned}
\tag{5.26}
$$

In the following, we give the first non-zero order of the dressing AFS phase. The kernel can be simply obtained by taking the derivative of the phase divided by 2π, with the right sign. We find, in the sectors " $<$ " and " $>$ ",

$$
\begin{aligned}
\theta^{>>}(u_1,u_2) &= \frac{2\epsilon}{\sinh\beta_1/2\sinh\beta_2/2}\,\frac{\sinh(\beta_1-\beta_2)/4}{\sinh(\beta_1+\beta_2)/4}-2\varphi^{>>}(u_1,u_2)+\mathcal{O}(\epsilon^2)\,,\\
\theta^{><}(u_1,u_2) &= -\frac{2\,\sin p_2/2}{\sinh\beta_1/2}-2\varphi^{><}(u_1,u_2)+\mathcal{O}(\epsilon)\,,\\
\theta^{<>}(u_1,u_2) &= \frac{2\,\sin p_1/2}{\sinh\beta_2/2}-2\varphi^{<>}(u_1,u_2)+\mathcal{O}(\epsilon)\,,\\
\theta^{<<}(u_1,u_2) &= \frac{1}{\epsilon}\Big[(\cos p_1/2-\cos p_2/2)\ln\frac{\sin^2(p_1-p_2)/4}{\sin^2(p_1+p_2)/4}\Big]+\mathcal{O}(1)\,.
\end{aligned}
\tag{5.27}
$$

In the sectors "$+$" and "$-$", where β and p (or $2\pi-p$) are of order $\sqrt{\epsilon}$, we introduce the parameter α, which is of order one, as follows:

$$
\begin{aligned}
p_i &= 2\sqrt{\epsilon}\,\alpha_i\,, \qquad \beta_i=2\sqrt{\epsilon}\,\alpha_i^{-1} \quad\text{with}\quad \alpha_i>0 \qquad \text{in the region" + "}\,,\\
p_i &= 2\pi+2\sqrt{\epsilon}\,\alpha_i, \quad \beta_i=-2\sqrt{\epsilon}\,\alpha_i^{-1} \quad\text{with}\quad \alpha_i<0 \qquad \text{in the region" − "}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\theta^{\pm\pm}(\alpha_1,\alpha_2) &= \pm\frac{1}{2}\Big[(\alpha_1^{-2}-\alpha_1^2-\alpha_2^{-2}+\alpha_2^2)\ln\frac{(\alpha_1^{-1}+\alpha_2^{-1})^2+(\alpha_1-\alpha_2)^2}{(\alpha_1^{-1}+\alpha_2^{-1})^2+(\alpha_1+\alpha_2)^2}\,,\\
&\qquad -4i\ln\frac{\alpha_1^{-1}+\alpha_2^{-1}-i(\alpha_1-\alpha_2)}{\alpha_1^{-1}+\alpha_2^{-1}+i(\alpha_1-\alpha_2)}\Big]+\mathcal{O}(\sqrt{\epsilon})\,,\\
\theta^{\pm\mp}(\alpha_1,\alpha_2) &= \pm 2\,\alpha_1\alpha_2+\mathcal{O}(\sqrt{\epsilon})\\
\theta^{\pm>}(\alpha_1,u_2) &= \pm 2\sqrt{\epsilon}\,\frac{\alpha_1}{\sinh\beta_2/2}+\mathcal{O}(\epsilon)\,,\\
\theta^{\pm<}(\alpha_1,u_2) &= -\frac{2}{\sqrt{\epsilon}}\,\alpha_1\sin p_2/2+\mathcal{O}(1)\,,\\
\theta^{>\pm}(u_1,\alpha_2) &= \mp 2\sqrt{\epsilon}\,\frac{\alpha_2}{\sinh\beta_1/2}+\mathcal{O}(\epsilon)\,,\\
\theta^{<\pm}(u_1,\alpha_2) &= \frac{2}{\sqrt{\epsilon}}\,\alpha_2\sin p_1/2+\mathcal{O}(1)\,.
\end{aligned}
\tag{5.28}
$$

In the elliptic parametrization, in the hyperbolic limit, we find for the regimes " > " and " < ":

$$\begin{aligned}
\theta^{>>}(s_1,s_2) &= 2\epsilon\sinh s\sinh s'\tanh\frac{s-s'}{2} - 2\varphi^{>>}(s_1,s_2)+\mathcal{O}(\epsilon^2) \\
\theta^{><}(s_1,s_2) &= -\frac{2\sinh s_1}{\cosh s_2} - 2\varphi^{><}(s_1,s_2)+\mathcal{O}(\epsilon) \\
\theta^{<>}(s_1,s_2) &= \frac{2\sinh s_2}{\cosh s_1} - 2\varphi^{<>}(s_1,s_2)+\mathcal{O}(\epsilon) \\
\theta^{<<}(s_1,s_2) &= \frac{2}{\epsilon}\left(\frac{\sinh(s_1-s_2)}{\cosh s_1\cosh s_2}\ln|\tanh\frac{s_1-s_2}{2}|\right)+\mathcal{O}(1)
\end{aligned} \tag{5.29}$$

In the regimes where one of the variables is in the region " + " or " − ", we put $\alpha_i=\pm e^{\pm s_i}$:

$$\begin{aligned}
\theta^{\pm\pm}(s_1,s_2) &= \mp\Big[(\sinh 2s_1-\sinh 2s_2)\ln\frac{1+e^{\pm 2(s_1+s_2)}\tanh^2\frac{s_1-s_2}{2}}{1+e^{\pm 2(s_1+s_2)}} \\
&\quad +4\arctan\left(e^{\pm(s_1+s_2)}\tanh\frac{s_1-s_2}{2}\right)\Big]+\mathcal{O}(\sqrt{\epsilon})\,, \\
\theta^{\pm\mp}(s_1,s_2) &= \mp 2e^{\pm(s_1-s_2)}+\mathcal{O}(\sqrt{\epsilon})\,,
\end{aligned} \tag{5.30}$$

$$\begin{aligned}
\theta^{\pm>}(s_1,s_2) &= 2\sqrt{\epsilon}\,e^{\pm s_1}\sinh s_2+\mathcal{O}(\epsilon)\,, \\
\theta^{\pm<}(s_1,s_2) &= \mp\frac{2}{\sqrt{\epsilon}}\,\frac{e^{\pm s_1}}{\cosh s_2}+\mathcal{O}(1)\,, \\
\theta^{>\pm}(s_1,s_2) &= -2\sqrt{\epsilon}\,e^{\pm s_2}\sinh s_1+\mathcal{O}(\epsilon)\,, \\
\theta^{<\pm}(s_1,s_2) &= \pm\frac{2}{\sqrt{\epsilon}}\,\frac{e^{\pm s_2}}{\cosh s_1}+\mathcal{O}(1)\,.
\end{aligned} \tag{5.31}$$

It is important to note that, in what concerns $\theta^{\pm\pm}$, the equation (5.30) is not the right answer, since in this region all the terms χ_n contribute to the $\mathcal{O}(1)$ term, as it was pointed out in [26]. It is interesting to note that $\theta^{\pm\pm}$ and $\theta^{\pm\mp}$ in eq. (5.30) depend naturally on the rotated coordinates

$$s=s_1+s_2\ ,\qquad \tilde{s}=s_1-s_2\ ,$$

and that for large values of s, positive or negative, we have

$$\theta^{++}(s_1,s_2)\simeq 2e^{s}(\tanh\frac{\tilde{s}}{2}-1-\sinh\tilde{s})\ .$$

It will be interesting to know whether such a structure is preserved by the full answer.

6 Solving the integral equations in the strong coupling limit with the dressing kernel

Once we obtained the leading term of the dressed kernels, we are able to compute the leading contribution to the energy of the states we are interested in. In the $su(1|1)$ and

$su(2)$ sectors, we do not need to obtain the functional form for the density explicitly. This is a fortunate situation, since we do not know yet the kernel in the $\pm, \pm$ region, where most of the roots are concentrated.

6.1 The $su(1|1)$ dressed case

The results of the computation by Beccaria *et al.* [33, 34] for the energy of the highest excited state in the $su(1|1)$ sector can be also derived from the integral equation. In the $su(1|1)$ sector, the total kernel is

$$\mathcal{K} = K + K_d \,. \tag{6.1}$$

First, we can show that the densities $\rho_<$ and $\rho_>$ are subdominant, while the main contribution comes from $\rho_\pm$. The density for the "giant magnons" region is given by the equation

$$\rho^< = \frac{1}{2\pi}\frac{dp}{ds} + \mathcal{K}^{<<}\rho^< + \mathcal{K}^{<>}\rho^> + \sum_{\pm}\mathcal{K}^{<\pm}\rho^\pm \,. \tag{6.2}$$

By symmetry, we have $\rho_+(\alpha) = \rho_-(-\alpha)$ and since

$$\int_0^\infty \alpha\rho_+(\alpha)d\alpha + \int_{-\infty}^0 \alpha\rho_-(\alpha)d\alpha = 0\,,$$

the $\pm$ terms in (6.2) cancel, we conclude that

$$\rho^< = \frac{1}{2\pi}\frac{dp}{ds} + \mathcal{K}^{<<}\rho^< + \mathcal{K}^{<>}\rho^> + \mathcal{O}(\sqrt{\epsilon}) \,. \tag{6.3}$$

All the terms except $\mathcal{K}^{<<}\rho^<$ are of the order 1. Since $\mathcal{K}^{<<} = \mathcal{O}(1/\epsilon)$ we deduce that[15] $\rho_<(s) = \mathcal{O}(\epsilon)$. Similarly, we can write

$$\rho^> = \frac{1}{2\pi}\frac{dp}{ds} + \mathcal{K}^{><}\rho^< + \mathcal{K}^{>>}\rho^> + \sum_{\pm}\mathcal{K}^{>\pm}\rho^\pm \,. \tag{6.4}$$

Again the leading term in the sum vanishes and all the terms in the r.h.s. are of the order ϵ, so $\rho^>(s)$ is also of order at most ϵ. Therefore most of the solutions of the Bethe equations concentrate in the regions $\pm$, or "near-flat space" region. In this region, we have

$$\rho^\pm = \frac{1}{2\pi}\frac{dp}{d\alpha} + \mathcal{K}^{\pm<}\rho^< + \mathcal{K}^{\pm>}\rho^>(s') + \mathcal{K}^{\pm\pm}\rho^\pm +' \mathcal{K}^{\pm\mp}\rho^\mp. \tag{6.5}$$

At the leading order, we are left with

$$\rho^\pm(\alpha) = \int d\alpha'\mathcal{K}^{\pm\pm}(\alpha,\alpha')\rho^\pm(\alpha') + \int d\alpha'\mathcal{K}^{\pm\mp}(\alpha,\alpha')\rho^\mp(\alpha') \,. \tag{6.6}$$

[15] It is proven in [24] that this kernel is not degenerate.

Specializing to the sign + and by integrating over α, this gives

$$k^+(\alpha) = -\frac{1}{2\pi}\int d\alpha'\theta^{++}(\alpha,\alpha')\rho^+(\alpha') - \frac{1}{2\pi}\int d\alpha'\theta^{+-}(\alpha,\alpha')\rho^-(\alpha')\,, \tag{6.7}$$

where $k^+(\alpha)$ is the counting function, $\rho^+(\alpha) = dk^+/d\alpha$, and the minus sign comes from the fact that $2\pi\,\mathcal{K}(\alpha,\alpha') = -\partial_\alpha\theta(\alpha,\alpha')$. By multiplication with $\rho(\alpha)$ and integration we get

$$\begin{aligned}\int_0^\infty d\alpha\; k^+(\alpha)\rho^+(\alpha) &= -\frac{1}{2\pi}\int_0^\infty d\alpha\int_0^\infty d\alpha'\theta^{++}(\alpha,\alpha')\rho^+(\alpha)\rho^+(\alpha') \\ &- \frac{1}{2\pi}\int_0^\infty d\alpha\int_{-\infty}^0 d\alpha'\theta^{+-}(\alpha,\alpha')\rho^+(\alpha)\rho^-(\alpha')\,.\end{aligned} \tag{6.8}$$

Due to the anti-symmetry of the phase θ^{++}, the first integral in the rhs vanishes, while the l.h.s. is equal to

$$\int_0^\infty d\alpha\; k^+(\alpha)\rho^+(\alpha) = \int_0^{1/2} dk^+\; k^+ = \frac{1}{8}\,. \tag{6.9}$$

Finally, we use that $\theta^{+-}(\alpha,\alpha') = 2\alpha\alpha'$ and the symmetry $\rho^+(\alpha) = \rho^-(-\alpha)$ to put the equation (6.8) in the form

$$\int_0^\infty d\alpha\;\alpha\;\rho^+(\alpha) = \sqrt{\frac{\pi}{8}}\,. \tag{6.10}$$

This is all what is needed to compute the leading order in the energy

$$E^d_{su(1|1)} = 4gL\int e^{-\beta/2}\sin p/2\;\rho(p)\;dp \simeq 4\sqrt{g}L\int_0^\infty d\alpha\;\alpha\;\rho^+(\alpha) = \sqrt{2\pi g}L\,. \tag{6.11}$$

6.2 The $su(2)$ dressed case

The leading behavoir of the dressed kernel in the $su(2)$ case is the same as the one in the $su(1|1)$ case. The only difference is that the number of magnons is $L/2$ for the antiferromagnetic state of a chain of length L. The maximum value of the counting function $k^+(\alpha)$ in (6.9) will be now 1/4. We therefore obtain

$$E^d_{su(2)} = \sqrt{2\pi g}M = \sqrt{\frac{\pi g}{2}}L\,, \tag{6.12}$$

which differs by a factor of $\sqrt{2}$ from the result obtained in [33, 34].

6.3 The $sl(2)$ dressed case

For the $sl(2)$ case, we are going to denote by $\mathcal{K}$ the total kernel without the $su(2)$ part which will need separate treatment

$$\mathcal{K}(u,u') = 2K(u,u') + K_d(u,u') \equiv \frac{1}{2\pi}\partial_u\phi(u,u')\,. \tag{6.13}$$

The BES equation, whose solution determins the dimension of the twist-two operator, includes the dressing phase:

$$\bar{\sigma}(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} du' \, \frac{2\epsilon\, \bar{\sigma}(u')}{(u-u')^2+4\epsilon^2} \; + \int du' \, \mathcal{K}(u,u')\, \bar{\sigma}(u') \;, \tag{6.14}$$
$$\text{with} \qquad \bar{\sigma}(u) \equiv \sigma(u) - \frac{2\epsilon}{\pi} \;.$$

The shifted density functon $\bar{\sigma}(u)$ is, up to a negative factor, the total density in the leading order in S, see (2.16). We are interested in expanding this equation in powers of ϵ, therefore we have

$$\frac{2\epsilon}{\pi} \fint_{-\infty}^{\infty} du' \, \left(\frac{\bar{\sigma}(u')}{(u-u')^2} + \mathcal{O}(\epsilon^2) \right) \; + \int du' \, \mathcal{K}(u,u')\, \bar{\sigma}(u') = 0 \;. \tag{6.15}$$

This equation breaks into several pieces which couple the different regions in u, *e.g.* if $|u|<1$ we obtain

$$\begin{aligned} &\frac{2\epsilon}{\pi}\left(\fint_{-1}^{1} du' \, \frac{\bar{\sigma}^{<}(u')}{(u-u')^2} + \int_{>} du' \, \frac{\bar{\sigma}^{>}(u')}{(u-u')^2} + \mathcal{O}(\epsilon^2) \right) + \int_{-1}^{1} du' \, \mathcal{K}^{<<}(u,u')\, \bar{\sigma}^{<}(u') \\ &+ \int_{>} du' \, \mathcal{K}^{<>}(u,u') \bar{\sigma}^{>}(u') + \sum_{\pm} \int d\alpha \, \mathcal{K}^{<\pm}(u,\alpha)\, \bar{\sigma}^{\pm}(\alpha) = 0 \;. \end{aligned} \tag{6.16}$$

The equation (6.14) was analyzed at the first two orders in ϵ by [24]. They did not consider the regions around $u=\pm 1$, where a fraction of roots can lie. The equation (6.14) suggests that the expansion of the density in ϵ starts at order ϵ. The structure of the leading terms in the expansion of the dressing kernel (5.28), where half-integer powers of ϵ appear, suggests that the density and the anomalous dimension may involve a half-integer powers of ϵ as well. However, these corrections seem to appear in higher orders in the anomalous dimension, and this may explain why they have not been seen yet[16]. In the following, we are going to ignore the corrections coming from around the points $u=\pm 1$ and solve perturbatively the equation (6.14), by considering the decomposition of the density and the kernel in integer powers of ϵ

$$\begin{aligned} \bar{\sigma}(u) &= \sigma(u) - \frac{2\epsilon}{\pi} = \epsilon\bar{\sigma}_1(u) + \epsilon^2\sigma_2(u) + \epsilon^3\sigma_3(u) + \ldots \\ \mathcal{K}(u,u') &= \epsilon^{-1}\mathcal{K}_{-1}(u,u') + \mathcal{K}_0(u,u') + \epsilon\mathcal{K}_1(u,u') + \ldots \end{aligned} \tag{6.17}$$

- **order** ϵ^0

At the leading order in ϵ the equation (6.16) reads, in simplified notations,

$$\mathcal{K}_{-1}\bar{\sigma}_2 + \mathcal{K}_0\bar{\sigma}_1 = 0.$$

[16] As far we can see, the first non-zero order for $\sigma^{\pm}(\alpha)$ is at most ϵ^2, and there may be corrections of order $\epsilon^{5/2}$ to $\sigma^{<}(u)$ and of order $\epsilon^{3/2}$ for $\bar{\sigma}^{>}(u)$. The last of these corrections would induce a term of order $g^{1/2}$ in the anomalous dimension. To decide whether they are here or not we have to go to higher orders in the expansion of the kernel.

To this order only the $<<$ sector contributes:

$$\int_{-1}^{1} du' \, \mathcal{K}_{-1}^{<<}(u,u') \, \bar{\sigma}_1^{<}(u') = 0 \; , \tag{6.18}$$

which implies $\bar{\sigma}_1^{<}(u) = 0$, or equivalently $\sigma_1 = 2/\pi$.

• **order** ϵ^1

$$\mathcal{K}_{-1}^{<<} \, \bar{\sigma}_2^{<} + \mathcal{K}_0^{<>} \bar{\sigma}_1^{>} = 0 \; . \tag{6.19}$$

The kernel in the second term is antisymmetric in $u' \to -u'$

$$\mathcal{K}_0^{<>}(u,u') = -\frac{1}{\pi}\partial_u \frac{\sqrt{1-u^2}}{\sqrt{1-u'^2}} \, \mathrm{sgn}\, u' \quad , \tag{6.20}$$

so that it vanishes upon integration against the symmetric function $\bar{\sigma}_1^{>}(u)$. Therefore, as pointed out in [24], the subleading density $\sigma_2^{<}(u)$ vanishes as well.

• **order** ϵ^2

$$\frac{2}{\pi} \fint_{-\infty}^{\infty} \frac{du' \bar{\sigma}_1(u')}{(u-u')^2} + \mathcal{K}_{-1}\bar{\sigma}_3 + \mathcal{K}_0\bar{\sigma}_2 + \mathcal{K}_1\bar{\sigma}_1 = 0 \, .$$

∘ The interval $|u| > 1$

Consider first the region $|u| > 1$, where the equation takes the form

$$\frac{2}{\pi} \fint_{>} du' \frac{\bar{\sigma}_1^{>}(u')}{(u-u')^2} + \int_{>} du' \, \mathcal{K}_1^{>>}(u,u') \, \bar{\sigma}_1^{>}(u') = 0 \, , \tag{6.21}$$

with the kernel $\mathcal{K}_1^{>>}(u,u') \equiv \frac{1}{2\pi}\partial_u \phi_1$ given by (5.29). The equation (6.21) is a total derivative, so we first integrate it to

$$\fint_{>} du' \bar{\sigma}_1^{>}(u') \left(-\frac{2}{\pi}\frac{1}{u-u'} + \frac{1}{2\pi}\phi_1(u,u') \right) = 0 \, . \tag{6.22}$$

As the phase is anti-symmetric, there is no integration constant.

At this point, we find it more useful to switch to the variable s, defined by $u = \coth s$, and (6.22) becomes

$$\frac{1}{\pi} \fint_{-\infty}^{\infty} ds' \, \bar{\sigma}_1^{>}(s') \sinh s' \, \coth \frac{s-s'}{2} = 0 \, . \tag{6.23}$$

Now we can reformulate the integral equation (6.22) as a Riemann boundary value problem. We first rewrite (6.23) in terms of the normalizable density $\sigma_1 = \bar{\sigma}_1 + 2/\pi$:

$$\frac{1}{\pi} \fint_{-\infty}^{\infty} ds' \, \sigma_1^{>}(s') \sinh s' \, \coth \frac{s-s'}{2} = \frac{4s}{\pi^2 \sinh s} \, . \tag{6.24}$$

Then, after the redefiniton

$$\omega = e^s, \quad r(\omega) = -r(1/\omega) = \sigma_1^>(s)\,\sinh s\,, \tag{6.25}$$

the l.h.s. of (6.24) takes the form of a Cauchy integral:

$$\int_0^\infty \frac{d\omega'}{\omega'} r(\omega') + 2 \,⨍_0^\infty d\omega' \frac{r(\omega')}{\omega - \omega'} = \frac{8}{\pi}\, \frac{\ln\omega}{\omega - \omega^{-1}}, \qquad (\omega > 0). \tag{6.26}$$

The first term on the l.h.s. is actually zero due to the anti-symmetry of $r(\omega)$. Eqn. (6.26) can be formulated in terms of a boundary condition for the resolvent

$$R(\omega) = R(1/\omega) = \int_0^\infty d\omega' \,\frac{r(\omega')}{\omega - \omega'}\,, \tag{6.27}$$

which has a cut on the positive axis, namely,

$$R(\omega + i0) + R(\omega - i0) \quad = \quad \frac{8}{\pi}\, \frac{\ln\omega}{\omega - \omega^{-1}} \qquad (\omega > 0)\,. \tag{6.28}$$

The most general solution of (6.28) with the symmetry $R(\omega) = R(1/\omega)$ is of the form

$$R(\omega) = \frac{4}{\pi} \frac{\ln(-\omega) + \sqrt{-\omega}\, Q(\omega) - \sqrt{-1/\omega}\, Q(1/\omega)}{\omega - 1/\omega}\,, \tag{6.29}$$

where $Q(\omega)$ is a rational function. The latter is determined by the analytical properties of the resolvent (no poles outside the positive real axis) and the requirement that that the density σ_1 is normalizable:

$$\int_> du \sigma_1(u) = \int_{-\infty}^\infty ds \sigma_1^>(s) = \int_0^\infty d\omega \frac{2r(\omega)}{\omega^2 - 1} = \text{finite}.$$

The only solution is $Q = -1$

$$R(\omega) \quad = \quad \frac{4}{\pi} \frac{\ln(-\omega) - \frac{1}{2}\left(\sqrt{-\omega}\, - \sqrt{-1/\omega}\right)}{\omega - 1/\omega}\,, \tag{6.30}$$

and its discontinuity along the positive real axis,

$$r(\omega) = \frac{R(\omega + i0) - R(\omega - i0)}{2\pi i} = \frac{4}{\pi} \left(\frac{1}{\omega - 1/\omega} - \frac{\sqrt{\omega}}{\omega - 1} \right)\,, \tag{6.31}$$

reproduces the solution found by Alday *et al* [24]:

$$\sigma_<(u) = \epsilon \sigma_1^< = \frac{1}{2\pi g}, \quad \sigma_>(u) = \epsilon\sigma_1^>(u) = \frac{1}{2\pi g}\left(1 - \cosh\frac{s}{2}\right). \tag{6.32}$$

The total integral of the density,

$$\int_{-\infty}^\infty ds\ (\sigma^<(s) + \sigma^>(s)) = \frac{1}{\pi g} + \frac{\pi - 4}{4\pi g} = \frac{1}{4g}\,, \tag{6.33}$$

reproduces the correct leading behavior for the twist-two anomalous dimension

$$f(g) = 4g + \dots \, . \tag{6.34}$$

∘ The interval $|u| < 1$

Let us assume for the moment that the corrections from the vicinity of the points $u = \pm 1$ can be neglected. Then, in the region $|u| < 1$, the equation reduces to

$$\frac{2}{\pi} \int_{|u'|>1} du' \, \frac{\bar{\sigma}_1^{>}(u')}{(u-u')^2} + \mathcal{K}_{-1}^{<<} \bar{\sigma}_3^{<} + \mathcal{K}_1^{<>} \bar{\sigma}_1^{>} = 0. \tag{6.35}$$

The kernels in the first and third term combine to

$$\mathcal{K}_1^{<>}(u,u') + \frac{2}{\pi} \frac{1}{(u-u')^2} = \frac{1}{\pi} \partial_u \, \frac{\sinh s' \cosh s + \frac{2}{\pi}(s-s') \sinh^2 s'}{\cosh(s-s')} \, . \tag{6.36}$$

This expression was obtained by expanding the first two terms χ_0 and χ_1 in the dressing phase, up to the desired order in ϵ. Integrating the product of the kernel (6.36) with $\bar{\sigma}_1^{>} = -2\cosh(s/2)$, gives zero, so we deduce that

$$\bar{\sigma}_3^{<}(u) = 0 \, . \tag{6.37}$$

• **order** ϵ^3

$$\frac{2}{\pi} \int_{-\infty}^{\infty} du' \bar{\sigma}_2(u')(u-u')^2 + \mathcal{K}_{-1}\bar{\sigma}_4 + \mathcal{K}_0 \bar{\sigma}_3 + \mathcal{K}_1 \bar{\sigma}_2 + \mathcal{K}_2 \bar{\sigma}_1 = 0 \, .$$

The subleading density $\sigma_2^{>}(u)$, which we need to compute the subleading term in $f(g)$, is supposed to be determined by the equation with $|u| > 1$:

$$\frac{2}{\pi} \int_{>} du' \, \frac{\sigma_2^{>}(u')}{(u-u')^2} + \mathcal{K}_1^{>>} \sigma_2^{>} = -\mathcal{K}_2^{>>} \bar{\sigma}_1^{>} \, . \tag{6.38}$$

After integrating with respect to u and passing to the s parametrization, this equation takes the same form as (6.24), but with different inhomogeneous term:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} ds' \sigma_2^{>}(s') \sinh s' \, \coth \frac{s-s'}{2} = -\frac{1}{2\pi \sinh s} \int_{-\infty}^{\infty} ds' \bar{\sigma}_1^{>}(s') \, \phi_2^{>>}(s,s') \, . \tag{6.39}$$

The inhomogeneous term in the r.h.s. can be computed from the expression

$$\phi_2^{>>}(s,s') = \frac{4(\sinh s \sinh s')^2}{\pi} \, \frac{(s-s')\cosh(s-s') - \sinh(s-s')}{\sinh^2(s-s')}, \tag{6.40}$$

which, after being integrated with $\bar{\sigma}_1^{>}(s) = -\frac{2}{\pi}\cosh(s/2)|u'(s)|$ gives

$$-\frac{1}{2\pi \sinh s} \int_{-\infty}^{\infty} \frac{ds'}{\sinh^2 s'} \, \sigma_1^{>}(s') \, \phi_2^{>>}(s,s') = -\frac{2}{\pi} \sinh s \sinh(s/2) \, . \tag{6.41}$$

After the redefinition (6.25) we again reduce the integral equation to a Riemann boundary value problem:

$$R(\omega + i0) + R(\omega - i0) = 2 \int_0^\infty d\omega' \frac{r(\omega')}{\omega - \omega'} = -\frac{\sqrt{\omega}(\omega - 1)(\omega^2 - 1)}{2\omega^2}\,. \tag{6.42}$$

The solution is

$$R(s) = -\frac{s \sinh s \cosh(s/2)}{2\pi}, \qquad r(s) = \frac{s \sinh s \sinh(s/2)}{2\pi^2},$$

which gives for the ϵ^2-correction to the density

$$\sigma_2^{>}(s) = \frac{s \sinh(s/2)}{2\pi^2}. \tag{6.43}$$

This correction to the density grows exponentially at $s \to \infty$ and is not normalizable. The result (6.43) means that the higher order corrections to the density become large in the vicinity of the points $u = \pm 1$, which correspond to $s = \pm\infty$. In order to determine the next correction to the scaling function $f(g)$ we have to take into account the contribution coming from the points $u = \pm 1$ (the regimes $\pm$). The equations starting with (6.35) should be corrected accordingly. We leave this problem for future work.

7 Outlook

We have presented here a method to take the strong coupling limit of the Bethe ansatz equations supposed to encode the spectrum of anomalous dimensions. We found it useful to express the spectral parameter appearing in the Bethe ansatz equations by means of a complexified momentum, with real part p and imaginary part β

$$x^{\pm} = e^{\beta/2 \pm ip/2} \tag{7.1}$$

It worths noticing that these variables already appeared in [31], where the BDS magnons were interpreted as bound states of more fundamental, fermionic excitations. There, p is the momentum of the bound states, while β is related to the "size" of the bound state.

Furthermore, the functions of p and β are naturally expressed in terms of elliptic functions. Our elliptic parametrization is a version of that of [9]. In the strong coupling limit, the elliptic parametrization degenerates into a hyperbolic one. In order to take into account properly the different regimes, we have to look to different expansions of the elliptic functions in terms of hyperbolic functions, depending on the value of the the elliptic parameter s: around $s = 0$, K and $\pm$K/2. These regimes correspond to the "plane-wave" [27], the "giant magnons" region [28, 29] and the "near-flat space" region recently characterized in [26], respectively. It is interesting to note that the momentum p and the energy $\varepsilon \equiv E/4g$ of the excitations have, in the plane wave and near-flat space regime, relativistic-like expressions

$$p = \epsilon \sinh s \qquad \varepsilon = \epsilon(\cosh s - 1) \qquad (\text{plane} - \text{wave}) \tag{7.2}$$

$$p = \pm 2\sqrt{\epsilon}\, e^{\pm s} \qquad \varepsilon = 2\sqrt{\epsilon}\, e^{\pm s} \qquad (\text{near} - \text{flat space}) \tag{7.3}$$

with s playing the role of the rapidity variable. However, only the positive branch of the energy, corresponding to particles, appear in the sectors we consider, and which should be closed at any order in perturbation theory. It is important to understand how the antiparticle branch will appear; most probably by another copy of the sector which joins in. An example of how this should happen is offered by the $su(2)$ principal chiral model [51, 52] , where the particles and antiparticles correspond to the two $su(2)$-symmetric excitations.

The strong coupling limit of the kernels without the dressing factor is relatively simple. The giant magnons interact through a delta term, typical for the statistical repulsion. In the $su(1|1)$ sector, there is also a "length-changing" term. The excitations of the plane-wave type do not interact with each other, at leading order, but they interact with the giant magnons. The excitations in the near-flat space regime do not contribute to the leading term.

The strong coupling limit of the dressing kernel is more involved. For the giant magnons and plane-wave regions, we can do it safely with the strong coupling expansion of the dressing kernel [10, 11, 12], at least for the first few orders in ϵ. For the near-flat space limit, however, as it was pointed out in [26], all the terms in the strong coupling series contribute to the leading order. We have given a integral representation of the dressing kernel, based on the "magic formula" in [13] which may be able to resum these contributions.

When we consider the kernels with the dressing phase, the repulsion of the giant magnons is so large, that their density vanishes in the leading ordrer. For $su(1|1)$ and $su(2)$ sectors, most of the magnons are concentrated in the near-flat space regime. Fortunately, although we do not have an explicit expression of the kernel in this regime, it is possible to obtain the energy for the highest excited state at leading order.

The most exciting application of our method is to obtain the strong coupling expansion of the twist-two operator anomalous dimension and to compare with the string computations [20, 21]. Several recent analytical and numerical works [22, 23, 24] were devoted to this task. In particular, [24] obtained the leading term, using the Fourier transform representation. They also tried to obtain the result by a method very similar to ours, and found that the first two orders of the equation are not sufficient to fix the leading order. Here, we show that it is possible to obtained the leading order, by going an order higher. We have also shwon that, in order to compute the higher correction, we have to take into account the contribution from the near-flat space regime.

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A The elliptic parametrization

The modulus of the elliptic map is

$$k = \frac{1}{\sqrt{1+\epsilon^2}}, \qquad k' = \frac{\epsilon}{\sqrt{1+\epsilon^2}}. \tag{A.1}$$

The real and imaginary parts of the complexified momentum

$$p(s) = 2k' \int_0^s \frac{dv}{\mathrm{dn}\, v}, \quad \beta(s) = \ln \frac{1+k'}{1-k'} - 2\int_K^s \mathrm{cs}\, v \, dv \tag{A.2}$$

have the symmetries

$$\begin{aligned} p(2K-u) &= 2\pi - p(u), \qquad \beta(2K-u) = \beta(u) \\ p(u) &= -p(-u), \qquad\quad \beta(-u) = \beta(u) \mp 2i\pi\,. \end{aligned} \tag{A.3}$$

The phase in the Bethe equations depends on β and p through

$$\begin{aligned} \cosh \tfrac{1}{2}\beta &= \frac{1}{k}\,\mathrm{ns}\, s, \quad \sinh \tfrac{1}{2}\beta = \frac{1}{k}\,\mathrm{ds}\, s, \\ \cos \tfrac{1}{2}p &= \mathrm{cd}\, s, \qquad \sin \tfrac{1}{2}p = k'\,\mathrm{sd}\, s. \end{aligned} \tag{A.4}$$

$$\begin{aligned} e^{\pm\beta/2} &= \frac{1 \pm \mathrm{dn}\, s}{k\,\mathrm{sn}\, s}, \quad e^{\pm ip/2} = \mathrm{cd}\, s \pm ik'\,\mathrm{sd}\, s \\ \partial_s p &= 2k'\mathrm{nd}\, s, \qquad \partial_s \beta = -2\,\mathrm{cs}\, s. \end{aligned} \tag{A.5}$$

Below we give asymptotic expressions of these functions in the limit $k' \simeq \epsilon \to 0$, where

$$K \approx \ln \frac{4}{\epsilon}, \quad K' \approx \frac{\pi}{2}, \quad \tilde{q} = e^{-\pi K/K'} \approx \frac{\epsilon^2}{16}. \tag{A.6}$$

In this limit it is appropriate to use the expansions of the three eliptic functions in terms of hyperbolic functions of $v = \pi s/2K'$:

$$\begin{aligned} \mathrm{sn}\, s &= \frac{\pi}{2kK'}\left(\tanh v + 4\sum_{n=1}^{\infty}(-)^n \frac{\tilde{q}^{2n}}{1+\tilde{q}^{2n}} \sinh 2nv\right) \\ \mathrm{cn}\, s &= \frac{\pi}{2kK'}\left(\frac{1}{\cosh v} + 4\sum_{n=1}^{\infty}(-)^n \frac{\tilde{q}^{2n-1}}{1+\tilde{q}^{2n-1}} \cosh(2n-1)v\right) \\ \mathrm{dn}\, s &= \frac{\pi}{2K'}\left(\frac{1}{\cosh v} - 4\sum_{n=1}^{\infty}(-)^n \frac{\tilde{q}^{2n-1}}{1-\tilde{q}^{2n-1}} \cosh(2n-1)v\right). \end{aligned} \tag{A.7}$$

In the limit $\epsilon \to 0$ and up to $o(\epsilon^4)$ terms the three functions are given by [41]

$$\mathrm{sn}\, s \;\simeq\; \tanh s \;+\; \epsilon^2 \frac{(\sinh s \cosh s - s)}{4\cosh^2 s}$$

$$
\begin{aligned}
\mathrm{cn}\, s &\simeq \frac{1}{\cosh s} - \epsilon^2 \frac{\tanh s}{4}\left(\sinh s - \frac{s}{\cosh s}\right) \\
\mathrm{dn}\, s &\simeq \frac{1}{\cosh s} + \epsilon^2 \frac{\tanh s}{4}\left(\sinh s + \frac{s}{\cosh s}\right).
\end{aligned} \tag{A.8}
$$

We used only the leading term; it gives a good approximation for $s \in I^{>} = [-\frac{1}{2}K, \frac{1}{2}K]$. Similarly, for the interval $s \in I^{<}$ we can use the approximation

$$
\begin{aligned}
\mathrm{sn}\,(\mathrm{K}-s) &\simeq \mathrm{cd}\, s \simeq 1 \\
\mathrm{cn}\,(\mathrm{K}-s) &\simeq k'\mathrm{sd}\, s \simeq \epsilon \sinh s \\
\mathrm{dn}\,(\mathrm{K}-s) &\simeq k'\mathrm{nd}\, s \simeq \epsilon \cosh s\,.
\end{aligned} \tag{A.9}
$$

For the functions that enter in the definition of the kernel we find:

(a) In the interval $I^{>}$:

$$
\begin{aligned}
\cosh(\beta/2) = \coth s, \quad \sinh(\beta/2) = \frac{1}{\sinh s}, \quad \beta' = -\frac{2}{\sinh s}, \\
\cos(p/2) = 1, \quad \sin(p/2) = k' \sinh s, \quad \partial_s p = 2k' \cosh s \\
u' = -\frac{1}{\sinh^2 s}, \quad u = \coth s \\
e^{\beta/2} = \coth \frac{s}{2}, \qquad e^s = \sqrt{\frac{u+1}{u-1}}
\end{aligned} \tag{A.10}
$$

(b) In the interval $I^{<}$, after replacing $s \to K - s$:

$$
\begin{aligned}
\cosh(\beta/2) = 1, \quad \sin(\beta/2) = k' \cosh s, \quad \partial_s \beta = -2k' \sinh s\,, \\
\cos(p/2) = \tanh s, \quad \sin(p/2) = 1/\cosh s, \quad \partial_s p = 2/\cosh s \\
u' = \frac{1}{\cosh^2 s}\,, \quad u = \tanh s, \\
e^{ip/2} = \frac{1+ie^{-s}}{1-ie^{-s}}\,, \qquad e^s = \sqrt{\frac{1-u}{1+u}}
\end{aligned} \tag{A.11}
$$

Plugging these expressions in (3.27) we evaluate the kernel in the four possible regimes. In the sector $s, s_1 \in I^{>}$ the numerator in (3.27) is of order ϵ, while the denominator remains finite. Therefore

$$
K^{>>}(s, s_1) = 0. \tag{A.12}
$$

For the non-diagonal elements of (3.29) we obtain

$$
K^{><}(s, s_1) = K(s, \mathrm{K} - s_1) \simeq \frac{1}{4\pi} \frac{\beta' \sin(p_1/2)}{\cosh(\beta/2) - \cos(p_1/2)}
$$

$$\simeq \frac{1}{2\pi}\,\frac{1}{\cosh(s-s_1)}\,,$$

$$K^{<>}(s,s_1) = K(\mathrm{K}-s,s_1) \qquad \simeq \frac{1}{4\pi}\,p' - \frac{1}{4\pi}\,\frac{p'\,\sinh(\beta_1/2)}{\cosh(\beta_1/2)-\cos(p/2)}$$
$$\simeq \frac{1}{2\pi}\,\frac{1}{\cosh s} - \frac{1}{2\pi}\,\frac{1}{\cosh(s-s_1)}. \qquad \text{(A.13)}$$

Finally, if both arguments are in $I^<$, then the kernel $K^{<<}(s,s_1) = K(\mathrm{K}-s,\mathrm{K}-s_1)$ vanishes except near the double pole of the denominator at $s = s_1$, where it can be approximated by a delta-function:

$$K^{<<}(s,s_1) = K(\mathrm{K}-s,\mathrm{K}-s_1) \qquad \simeq \frac{1}{4\pi}\,p' - \frac{1}{2\pi}\,\frac{p'\,\beta}{\beta^2+\sin^2(p-p_1)/2}$$
$$\simeq \frac{1}{4\pi}\,p' - |\partial_s p|\,\delta\,(p-p_1)$$
$$\simeq \frac{1}{2\pi}\,\frac{1}{\cosh s} - \delta(s-s_1). \qquad \text{(A.14)}$$

- *The regime $s \simeq \pm K/2$*

Assume that $s = \pm K/2 + y$, where $y \ll \sqrt{\epsilon}$. We first evaluate the three basic Jacobi elliptic functions for the shifted argument:

$$\begin{aligned}
\frac{\mathrm{sn}\,(s\pm\frac{1}{2}\mathrm{K})}{\mathrm{sn}\,\frac{1}{2}\mathrm{K}} &= \frac{k'\mathrm{sd}\,\pm\mathrm{cn}}{\mathrm{cn}^{\,2}+k'\mathrm{sn}^{\,2}}(y)\\
\frac{\mathrm{cn}\,(s\pm\frac{1}{2}\mathrm{K})}{\mathrm{cn}\,\frac{1}{2}\mathrm{K}} &= \frac{\mathrm{cn}\,\mp\mathrm{sn}\,\mathrm{dn}}{\mathrm{cn}^{\,2}+k'\mathrm{sn}^{\,2}}(y)\\
\frac{\mathrm{dn}\,(s\pm\frac{1}{2}\mathrm{K})}{\mathrm{dn}\,\frac{1}{2}\mathrm{K}} &= \frac{\mathrm{dn}\,\mp(1-k')\mathrm{sn}\,\mathrm{cn}}{\mathrm{cn}^{\,2}+k'\mathrm{sn}^{\,2}}(y)\,.
\end{aligned} \qquad \text{(A.15)}$$

When $k' \approx \epsilon \to 0$, the values of the three functions at $s = \frac{1}{2}\mathrm{K}$ are

$$\begin{aligned}
\mathrm{sn}\,(\tfrac{1}{2}\mathrm{K}) &= 1/\sqrt{1+k'} \approx 1-\tfrac{1}{2}\epsilon,\\
\mathrm{cn}\,(\tfrac{1}{2}\mathrm{K}) &= \sqrt{k'/(1+k')} \approx \sqrt{\epsilon},\\
\mathrm{dn}\,(\tfrac{1}{2}\mathrm{K}) &= \sqrt{k'} \approx \sqrt{\epsilon},
\end{aligned} \qquad \text{(A.16)}$$

so that

$$\begin{aligned}
\mathrm{sn}\,(\pm\tfrac{1}{2}\mathrm{K}+y) &= \pm(1-\tfrac{1}{2}\epsilon\,e^{\mp 2y})\\
\mathrm{cn}\,(\pm\tfrac{1}{2}\mathrm{K}+y) &= \sqrt{\epsilon}\;e^{\mp y}\left(1-\epsilon\sinh^2 y\right)\\
\mathrm{dn}\,(\pm\tfrac{1}{2}\mathrm{K}+y) &= \sqrt{\epsilon}\;e^{\mp y}\,(1\pm\epsilon\sinh y\cosh y)
\end{aligned} \qquad \text{(A.17)}$$

From here one finds for the asymptotics of the functions $p(s)$, $\beta(s)$ and $u(s)$ for $s = \pm \mathrm{K}/2 + y$:

$$\begin{aligned}
\sinh(\beta/2) &= \pm\sqrt{\epsilon}\, e^{\mp y} \left(1 + \tfrac{1}{2}\epsilon \cosh 2y\right) \\
\sin(p/2) &= \pm\sqrt{\epsilon}\, e^{\pm y} \left(1 - \tfrac{1}{2}\epsilon \cosh 2y\right) \\
u &= \pm 1 - \epsilon \sinh 2y + o(\epsilon^2).
\end{aligned} \tag{A.18}$$

B Evaluation of K_d^0

Here we will evaluate the integral (5.21) for the strong coupling limit of K_d. We have

$$\begin{aligned}
K_-(x_1, x) &= \frac{K_-^p dp_1 + K_-^\beta d\beta_1}{4\pi} \\
K_+(x, x_2) &= \frac{K_+^p dp + K_+^\beta d\beta}{2\pi} \\
dz &= \tfrac{1}{2} Z^p dp + \tfrac{1}{2} Z^\beta d\beta,
\end{aligned} \tag{B.1}$$

where

$$\begin{aligned}
K_-^p &= \frac{\cos(p - p_1) - e^{-|\beta+\beta_1|}}{H_1}, \quad K_-^\beta = -\frac{\sin(p_1 - p)}{H_1}\,; \\
K_+^p &= \frac{\sinh\frac{\beta+\beta_2}{2} \cos\frac{p-p_2}{2}}{H_2}, \quad K_+^\beta = -\frac{\sin\frac{p-p_2}{2} \cosh\frac{\beta+\beta_2}{2}}{H_2}\,; \\
Z^p &= \sinh(\beta/2)\cos(p/2), \qquad Z^\beta = \sin(p/2)\cosh(\beta/2)\,; \\
H_a &= \cosh(\beta + \beta_a) - \cos(p - p_a), \quad a = 1, 2\,.
\end{aligned} \tag{B.2}$$

The integral to evaluate is

$$\begin{aligned}
K_d^0(x_1, x_2) &= -\frac{1}{2\pi^2\epsilon} \int_0^{2\pi} dp \int_0^\infty d\beta\; (K_-^p dp_1 + K_-^\beta d\beta_1)(Z^p K_+^\beta - Z^\beta K_+^p) \\
&= \frac{1}{4\pi^2\epsilon} \int_0^{2\pi} dp \int_0^\infty d\beta\; [A(p,\beta) dp_1 + B(p,\beta) d\beta_1]\,,
\end{aligned} \tag{B.3}$$

where

$$\begin{aligned}
A(\beta, p) &= (\cos(p - p_1) - e^{-|\beta+\beta_1|})\; G(\beta, p), \\
B(\beta, p) &= -\sin(p_1 - p)\; G(\beta, p), \\
G(\beta, p) &= \frac{\sin(p - \frac{p_2}{2}) \sinh(\beta + \frac{\beta_2}{2}) + \sinh\frac{\beta_2}{2} \sin\frac{p_2}{2}}{(\cosh(\beta + \beta_1) - \cos(p_1 - p))\,(\cosh(\beta + \beta_2) - \cos(p - p_2))}\,.
\end{aligned} \tag{B.4}$$

The strategy of calculation is the following: first we take an integral over p, which is of the form:

$$I = \int_0^{2\pi} dp\, R(\cos p, \sin p), \tag{B.5}$$

where R is a rational function. By symmetrization $p \to -p$ we reduce the integral to

$$I = \int_0^{2\pi} dp\, \tilde{R}(\cos p, \sin^2 p) \tag{B.6}$$

and, after substituting $\cos p = -t$, it can be evaluated by taking the residues.

The tntegral over β becomes, after the substitution $\beta = -\log x$, an integral from 0 to 1 of a rational expression and can also be performed explicitly. As a final result, we obtain

$$\int_0^\infty d\beta \int_0^{2\pi} dp\, A(p,\beta) = i\pi \left(\frac{1}{x_2^-} + \frac{1}{2}(x_1^+ - \frac{1}{x_1^+}) \log \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 + \frac{1}{x_1^+ x_2^-}} - \text{c.c.} \right), \tag{B.7}$$

$$\int_0^\infty d\beta \int_0^{2\pi} dp\, B(p,\beta) = \pi \left(\frac{1}{x_2^-} + \frac{1}{2}(x_1^+ - \frac{1}{x_1^+}) \log \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 + \frac{1}{x_1^+ x_2^-}} + \text{c.c.} \right). \tag{B.8}$$

From here we deduce the expression for K_d^0, presented in (5.22).

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